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INDIAN AGRICULTURAL
RESEARCH INSTITUTE, NEW DELHI.

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MGIPC—S4—51 AR/57—3-4-58—5,000.

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL LVI part I-II 1935-36.

I—The Lunar Atmospheric Tide at Glasgow. By Professor S. Chapman, F R.S. *Communicated by* R. A. Robb, M Sc

(MS. received November 12, 1935. Read December 2, 1935)

1. On seeing in *Nature* (June 29, 1935) a summary of the paper "The Lunar Atmospheric Pressure Inequalities at Glasgow," recently published in these *Proceedings* by Robb and Tannahill (1935), I wrote to Mr Robb (with whom I had had correspondence and discussion on the same subject in 1926) to point out that a certain "convex effect" discovered by Bartels was probably the main cause of their results It was fortunately not too late for the authors to refer to this point by adding two sentences at the end of their paper. But despite this addition I fear that their paper will produce a wrong impression on almost all its readers, particularly since no reference was given to Bartels' paper. It seems, therefore, desirable to state that the inequalities described in their paper are not lunar in any real sense. This is strictly true for the major (diurnal) component of the inequalities, and substantially true for the semidiurnal component. It is of some interest and importance to demonstrate this to the many readers of the paper who will not have previous acquaintance with the subject or ready access to its literature.

2. The inequalities were determined by a process which in effect amounts to the calculation of the average form of a number of "sections" of the barograph record, each section commencing at a solar hour closely following—*i.e.* just succeeding—a lunar upper transit, and covering a period of 25 solar hours, practically equal to one lunar day. If the whole of the record had been treated in this way, the averaged curve would represent the true lunar variation of the pressure, provided the record was long enough to average out the regular solar daily variation

and the accidental or irregular variations. The latter, however, are so large, compared with the lunar variation, that they can be averaged out only by means of a very long series of records—longer than is afforded even by the fine series of observations at Glasgow, extending from 1868 to 1912. Hence Robb and Tannahill followed the practice, first introduced in my discussion of the Greenwich barometric data, of discarding most of the record, and retaining only the "quiet" sections in each of which the range of pressure did not exceed 0·10 inch of mercury.

Such quiet sections of the record do not occur equally at all atmospheric pressures, it is a property of the variation of the pressure that when the barometer is high its changes are on the average slower and more gradual than when the level is low. Hence the quiet sections are drawn predominantly from anticyclonic periods when the barometer is high, or above its average level. At such times the curve is near a maximum value, and is therefore convex. This convexity may of course be combined with a small upward or downward slope, which is likely to average out nearly to zero in the mean of many days. The convexity may not appear in every individual quiet section of the record, but it will be evident in the average of all the sections. The inequalities found by Robb and Tannahill show it clearly, and it constitutes the main part of their inequalities. This is true of both their uncorrected and corrected inequalities: the corrections embodied in the latter depend on the solar daily variation of pressure and of vapour pressure, derived from *all days* (45-year normals); of course, these neither embody nor correct for the convexity characteristic of quiet periods.

The convexity does not depend on the precise length of the "sections," nor on the relation of the initial time of each section to the lunar transit; it would show in a similar set of sections of the same duration and range, commencing at any particular *solar* hour—such as midnight—or commencing at quite random times. The convexity would appear also if the sections were of different length or duration, provided that the range-limit was altered along with the duration.

3. When harmonically analysed, the average convexity, if simple and symmetrical with respect to the middle of the section, gives a main term of frequency 1 in the duration of the section, and of phase angle 270°: and smaller sub-harmonics, of frequencies 2, 3, . . ., and of amplitudes diminishing as the frequency increases, but all with the same phase angle 270°. The presence (in the inequalities) of some accidental variation not fully averaged out might, of course, modify these relations slightly. This is illustrated in the analysis of the inequalities found by Robb and Tannahill (Table IV); the first three components, of lunar

diurnal, semidiurnal, and terdiurnal type, have decreasing amplitudes 64, 15·6, and 8·9 microbars, and phases 267°, 285°, 292°, all near 270°.

This explains why such harmonic components necessarily arise in "lunar" daily inequalities obtained as described in § 2; and it is clear that they have no real connection with the moon. The concordance between the analyses by Robb and Tannahill of different subdivisions of their data (Table IV) is merely a consequence of the regularity with which convexity is shown by the barograph curve during quiet periods. Their test of "significance" (p. 93) bears only on the same point, and not on the lunar character of the variation. The authors say, with truth, (p. 96) that the large diurnal component, having its maximum at the lower transit of the moon, cannot be considered as accidental; but if they had taken the sections representing the lunar day to begin at or near lunar lower transit instead of lunar upper transit, they would certainly have found a similar diurnal component with its maximum in the middle of the section, *i e* at lunar *upper* transit, and therefore of phase opposite to that given in their paper. There is thus no evidence in *favour* of regarding this diurnal variation as lunar in origin, and there is every theoretical reason to believe that there is no appreciable real lunar diurnal variation of the pressure (in the average of many whole lunations). The same applies to the term of frequency 3 in the lunar day.

4. The semidiurnal component found by Robb and Tannahill is likewise mainly an artificial and non-lunar consequence of the convexity of the quiet-day barographs; but their inequalities should include also the real lunar semidiurnal pressure inequality due to the lunar atmospheric tide. To judge by the results found at stations elsewhere, this is likely to have a phase differing by not more than 30° or 40° from 90°. Thus the presumption is that the part of the semidiurnal component due to the convexity is actually somewhat greater than the component $15{\cdot}6 \sin(2t+285°)$ found by Robb and Tannahill. In a paper on the lunar atmospheric tide at Buenos Ayres, Chapman and Austin (1934) showed how to estimate the semidiurnal component due to convexity, assuming that the convex curve is a parabolic arc, which may not be strictly true: this method gives $15{\cdot}3 \sin(2t+270°)$ for the convexity in the last column of Table II (Robb and Tannahill). Subtracting this from their value of the semidiurnal variation of pressure, the true lunar variation at Glasgow (1868–1912) is found to be approximately $8 \sin 2t$ microbars. This result is, of course, subject to accidental error due to irregular causes. Neither the amplitude nor the phase angle (zero) are likely to be well determined, on account of the small amplitude and the large probable

error, estimated by Robb and Tannahill as 3 microbars in amplitude, or 12° in phase (both these estimates are probably too small).

5. In 1926 Mr Robb consulted me about an earlier reduction of the Glasgow data that he had then already made; he had followed the method that in 1918 I had adopted for Greenwich, not only in restricting himself to quiet days, but also in using sections of the barograph consisting of *solar* days (from one midnight to the next), and, in the lunar re-tabulations, transposing the part of the solar day that occurred before lunar transit on that day, so as to follow the part that preceded the transit. This transposition (fortunately, though not designedly) eliminated the convexity effect (the importance of which, in the determination of the lunar atmospheric tide, was described by Bartels in 1927). Mr Robb's result for the semidiurnal component, derived from 4358 selected quiet days at Glasgow, 1868–1912, was

$$4{\cdot}7 \sin (2t+83^\circ)$$

His values of c_2 and θ_2 for various subdivisions of his data were, however, somewhat discordant, namely:

	c_2	θ_2
1868–1882	6	-33°
1883–1897	14	89°
1898–1912	4	130°
Equinox and winter	8	74°

On this account he and I both regarded the results as unsatisfactory. I found that in correcting for the solar diurnal variation he had used the mean solar diurnal variation from all days of the 45-year period; I suggested that he should revise the correction by using the solar daily variation from the actual quiet days He did so, but the results, as follows, were very little altered:

	c_2	θ_2
1868–1882	7	-34°
1883–1897	14	89°
1898–1912	6	96°
Summer	2	183°
Equinox and winter	9	73°
Total	4·8	82°

This work, at least, confirmed that the method of transposition effectively eliminated the convexity of the quiet-day barograph curve. Mr Robb also allowed for the discontinuity introduced by the transposition, and found $5{\cdot}3 \sin (2t+53^\circ)$ as the corrected result for the whole period: this does not differ significantly from $4{\cdot}7 \sin (2t+83^\circ)$.

In 1926 I recommended Mr Robb to publish these results, but he decided to make yet another discussion of the Glasgow data, in the hope of gaining more consistent results. He has now done this with the help of Mr Tannahill, but has unfortunately encountered the pitfall of the convexity effect, unsuspected either by him or by me in 1926. It now seems clear to me, however, that his 1926 results are right and valuable to this extent, that they show the lunar atmospheric tide at Glasgow to be very small, of amplitude 4 or 5 microbars (whereas at Greenwich and Aberdeen the amplitudes are respectively 12 and 15 microbars) Being small, and being affected by probable errors of perhaps 2 or 3 microbars, comparable with, though perhaps rather smaller than the whole amplitude, the phase angle is somewhat uncertain, though it may be expected to be between about 50° and 90° The seasonal results, and the results for the sub-groups of years, are presumably discordant merely because of the still greater probable error which affects them

Mr Robb has kindly allowed me, in this discussion, to mention his 1926 results, which he had at that time communicated to me in manuscript. They entitle him to the credit of having made a real and valuable determination of the lunar atmospheric tide at Glasgow, which renders it probable that the amplitude there is abnormally small for the latitude (as I have found to be the case also for Victoria and Vancouver in Canada).

I am glad to be able to add, with the permission of Mr Robb and Mr Tannahill, that they are further investigating this matter, and particularly the diurnal component of the pressure variation.

SUMMARY

Reasons are given for believing that the lunar hourly inequalities of barometric pressure at Glasgow recently published by Robb and Tannahill are substantially not of lunar origin at all. An earlier determination of the lunar atmospheric tide at Glasgow made in 1926 by Mr Robb is, however, now published with his permission, and appears to be a true lunar effect. It indicates that at Glasgow the lunar atmospheric tide is abnormally small for the latitude, the phase seems normal, and corresponds to a lag of high tide on the atmosphere of less than one hour after lunar transit.

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(*Issued separately March* 17, 1936.)

II—The Effect of Present Trends in Fertility and Mortality upon the Future Population of Scotland and upon its Age Composition.

By **Enid Charles,** M.A., Ph.D. (From the Department of Social Biology, University of London.) *Communicated by* Professor LANCELOT T. HOGBEN, D.Sc. (With Seven Figures.)

(MS. received October 12, 1935. Read December 2, 1935.)

1. INTRODUCTION

IN a previous paper ("The Effect of Present Trends in Fertility and Mortality upon the Future Population of England and Wales and upon its Age Composition," *London and Cambridge Economic Service Memorandum No.* 40, 1935) the author prepared estimates of the future population of England and Wales on the basis of present trends in fertility and mortality. The present contribution gives estimates of the future population of Scotland based on similar considerations. For a full description of the significance of the methods employed, the reader is referred to the previous memorandum.

Although the net reproduction rate in Scotland in recent years has been higher than that in England and Wales, there has been a large outward balance of migration, resulting in a decrease of about 40,000 in the Census population of 1931 as compared with that at the Census of 1921. More recently the balance of migration has been inwards, resulting in an increasing population. The population figures upon which the present estimates are based are those for mid-year 1934, obtained from the Registrar-General for Scotland, to whom acknowledgments are due. They were adjusted to January 1935 by the method described in the previous paper, and from that date onwards the estimates take no account of emigration or immigration.

The specific death-rates used were those for 1933. Life tables, constructed by the abridged method on the basis of these rates, resulted in an expectation of life for males of 56·66 years and for females of 60·32. These may be compared with the Registrar-General's Life Tables, 1934, based on the Census of 1931 and on mortality for 1930, 1931, and 1932, which give the expectation of life for males and females as 56·0 and 59·5 respectively.

The specific fertility rates used were those for 1934. They were obtained by adjusting the births according to the age distribution of fertility in Sweden in 1926. The gross reproduction rate thus obtained was 1·098 and the corresponding net reproduction rate ·912 Estimate (*a*) is thus based on the fertility of 1934 and the mortality of 1933, and these rates are assumed to remain constant thereafter In 1921 the gross reproduction rate of Scotland was 1·545, and in 1931 it was 1·18. This is a drop of 23·6 per cent The drop since 1931 has been rather steeper. A consideration of the rates of change of the gross reproduction rates in England and Wales and in Scotland made it appear that the change was sufficiently similar in both countries to justify the application to Scotland of the same assumptions as to future rate of fall as those used for England and Wales. In estimate (*b*), therefore, the same assumptions are made about the future fall in fertility and mortality rates as were made for estimate (*b*) for England and Wales The populations in future years are calculated by the quinquennial method.

2. Results.

The population by quinquennial age groups for every fifth year from 1935 to 2035 was computed according to both estimates (*a*) and (*b*). The tables embodying these computations, together with those on which

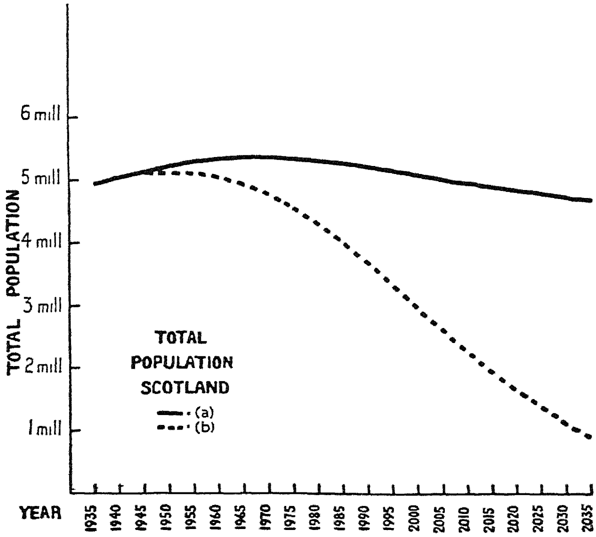

FIG. 1.—Total population of Scotland, 1935–2035, calculated according to two different assumptions

	1935	2035
(*a*)	4,945,000	4,647,000
(*b*)		925,000

later graphs are based, have been omitted for reasons of economy. The tables in full have been placed in the Archives of the Royal Society of Edinburgh, where they may be consulted.

The total populations according to both estimates are shown in the form of a graph in fig. 1. The graph shows that the present rate of reproduction in Scotland is sufficiently high to maintain the population at a practically stationary level for a very long time. On the other hand, if the rate of fall seen in past years were to continue, it would eventually result in a steeply declining population.

3. AGE COMPOSITION.

As in the case of England and Wales, the changes in the age composition of the population, which are taking place as a result of the falling fertility of past years or which might take place as a result of a

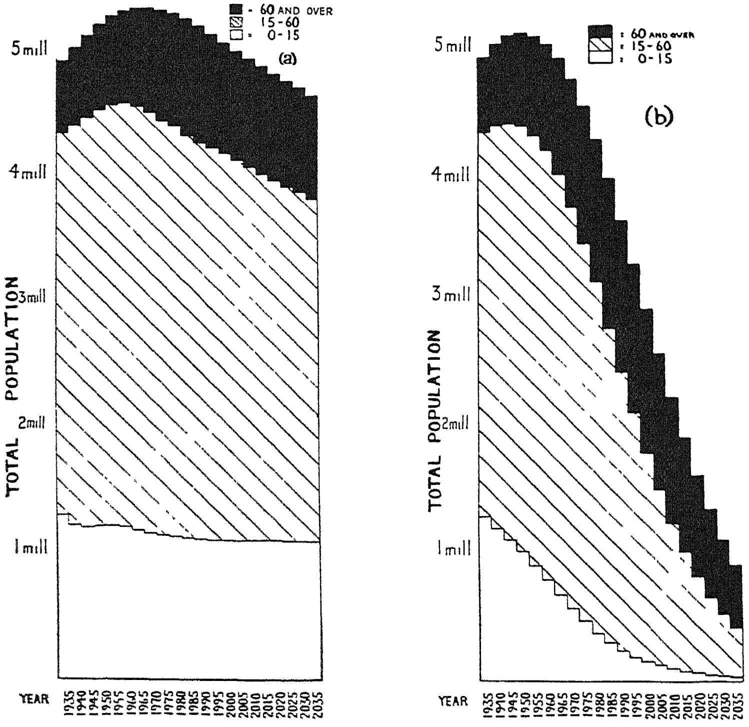

FIG 2.—Total numbers in the age groups 0–15, 15–60, 60 and over according to estimates (*a*) and (*b*), 1935–2035.

	1935	2035 (*a*)	2035 (*b*)
0–15	1,306,000	990,000	34,000
15–60	3,054,000	2,837,000	393,000
60+	584,000	820,000	498,000

future fall, are of more immediate interest than changes in the size of the total population. The percentage age composition of the population from 1935 to 2035 was computed

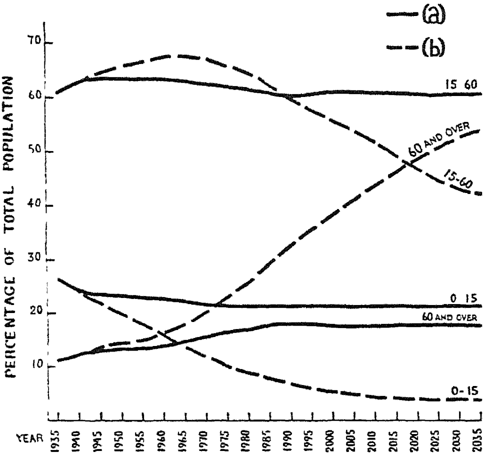

FIG 3—Percentage of the total population in the age groups 0–15, 15–60 60 and over according to estimates (*a*) and (*b*), 1935–2035.

	1935	2035	
		(*a*)	(*b*)
0–15	26 4	21·3	3·6
15–60	61 8	61 0	42 5
60 +	11 8	17 7	53 8

The totals on which the percentages are based are given for the three larger age groups in the form of histograms in fig. 2. Graphs of the percentages in these groups are shown in fig 3.

With fertility and mortality constant at their present level there is practically no change in the proportion of people aged 15–60. The percentages of children and old people approach more nearly to each other, but do not change places as they do in England and Wales in the same circumstances The graph of age composition in conditions of declining fertility and mortality is very similar to that for England and Wales and shows the same steeply rising percentage of old people.

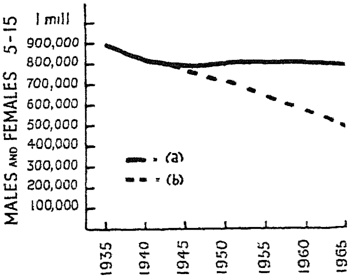

FIG 4—Total numbers in the age group 5–15, 1935–1965, estimates (*a*) and (*b*)

As the population was not calculated by single years, no details of educational groups can be given, but as the numbers of children aged 5–15 correspond fairly nearly to the elementary school population, the totals at these ages are shown graphically in fig. 4. The effect of the declining fertility of past years is shown in a rapid decline in the school population during the next ten years. On the assumption that fertility continues to fall, this decline would also continue. In estimate (*a*) the effect of constant fertility is to produce a practically stationary school population between 1945 and 1965, after which it would begin to decline slowly.

4 CRUDE BIRTH- AND DEATH-RATES.

The crude birth- and death-rates for the total population are shown graphically in fig. 5. The graph shows the rising death-rate resulting from an ageing population. In fig. 5 the point at which the birth- and death-rate curves cross indicates the point at which the population reaches its maximum, and the vertical distance between the two curves indicates the rate of decrease of the population.

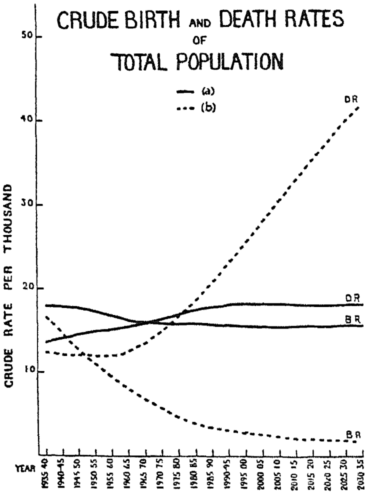

FIG 5.—Crude annual birth- and death-rates per thousand of the total population, 1935–2035, estimates (*a*) and (*b*)

	1935–39		2030–34	
	(*a*)	(*b*)	(*a*)	(*b*)
B R	18 0	16·8	15 6	1·8
D R.	13 6	12·5	18 2	41 9

5. THE SEX RATIO.

The ratio of females to males in different age groups and in the total population according to both estimates is shown in figs. 6 and 7. The curves closely resemble those given for England and Wales. The principal difference seen is that, owing to higher mortality in Scotland, there are relatively fewer females in the older age groups and hence the numbers of males and females in the total population tend to become more nearly equal.

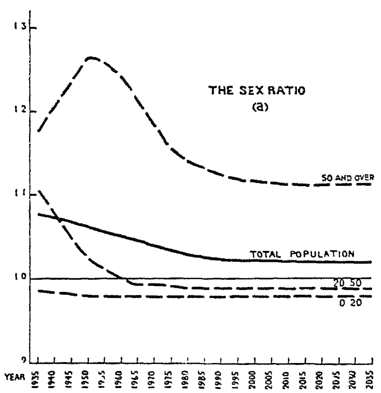

Fig 6 —The sex ratio (females to males) in the age groups 0–20, 20–50 50 and over and in the total population, 1935–2035, estimate (*a*).

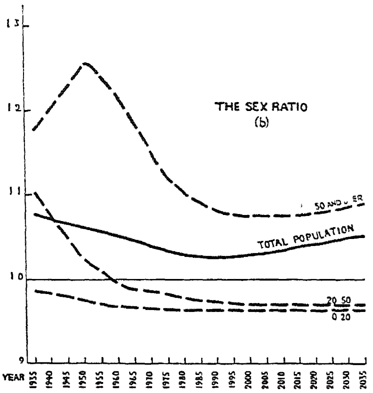

Fig 7 —The sex ratio (females to males) in the age groups 0–20, 20–50, 50 and over and in the total population, 1935–2035, estimate (*b*)

6 Summary.

1. Estimates of the population of Scotland during the next hundred years have been made on the following assumptions:—

(*a*) That fertility and mortality for each year of age remain constant at the levels of 1934 and 1933 respectively.

(*b*) That fertility and mortality continue to fall in the manner suggested by the figures available for the last decade.

2. Excluding the effect of emigration or immigration, the population of Scotland will begin to decline in numbers about 1970 according to estimate (*a*) and about 1950 according to estimate (*b*). According to (*a*) it will not fall below its present level until the period 2010–2015.

3. According to (*a*) a long period during which the population would be practically stationary would be followed by a slow decline. At the end of a hundred years the population would be 94 per cent. of its present size According to (*b*) a period of rapid decline would set in after about thirty years. In this estimate the population would be 81 per cent. of its present size in fifty years from now, and 19 per cent. of its present size a hundred years from now.

4 Apart from any further fall in fertility, the effect of the declining fertility of the past few years will be seen in a decreasing proportion of children and an increasing proportion of aged persons in the population

According to estimate (*a*) the age group 5–15 will decrease to 90 per cent. of the present total twenty years from now. According to estimate (*b*) it will decrease to 72 per cent The age group of 60 and over will increase to 123 per cent. of the present figure twenty years from now according to estimate (*a*). According to estimate (*b*) it will increase to 133 per cent

5 Twenty years from now the age group 5–15 will constitute 15·3 per cent. of the total population according to estimate (*a*) and 12·7 per cent. according to estimate (*b*). The age group of 60 years and over will constitute 13·5 per cent. of the population according to estimate (*a*) and 15·2 per cent according to estimate (*b*)

6. In 1935 the percentage of persons between 15 and 60 years of age in the population was 61·8 per cent. According to estimate (*a*) it will be 63·3 per cent. in 1955 and 60·7 in 1985. According to estimate (*b*) it will be 66·7 in 1955 and 62·7 in 1985.

7. The changing age composition of a declining population is likely to result in a rising crude death-rate in spite of lower specific death-rates. On estimates (*a*) and (*b*) the crude birth-rates per thousand fifty years from now will be 15·7 and 3·35 respectively, and the death-rates 17·87 and 20·73 respectively.

8. The present excess of females due to war losses will soon disappear. There will be an excess of males in the age group 20–50 by 1960 according to both estimates. The older age groups will continue to show an excess of females over males.

9. According to the first alternative adopted, which assumes constant fertility and mortality from now on, a nearly stable age composition and annual rate of decrease would be reached in about sixty years. The annual rate of decrease at the end of a hundred years would be ·26 per cent. per annum. According to the second alternative, the annual rate of decrease at the end of a hundred years would be 4 per cent. per annum.

(*Issued separately March* 17, 1936)

III —Studies in Clocks and Time-keeping: No. 5. The Suspended Chronometer. By Professor R. A. Sampson, LL.D, F R.S. (With Four Figures)

(MS. received November 20, 1935 Read January 13, 1936)

IN his *Popular Lectures and Addresses*, vol. ii, 1894, p 360, Lord Kelvin published a paper which he had read before the Institution of Engineers in Scotland in 1867, on "The Rate of a Clock or Chronometer as influenced by the Mode of Suspension." The effect has long been known. Almost two centuries ago, Ellicott observed the marked beats exchanged by two similar regulators when their cases were connected, which is due to the same effect (*Phil. Trans*, vol. xli, 1744, p. 126). The fact has crystallised into the rule that the cock that holds the pendulum ought to be as firmly fixed as possible; but as the question is not familiar, and as the observations of Lord Kelvin are given in general terms, and his description—though remarkably correct—contains some oversights, I thought it worth while to repeat the theory and the experiment.

I shall give the latter first. The chronometer used was a sidereal chronometer, marked "171" without any maker's name. It had held a gaining rate of $-3^{s}{\cdot}25$ per day during the ten weeks before the experiment, the extreme values being $-3^{s}{\cdot}1$ and $-3^{s}{\cdot}6$. This is about the best one expects from a chronometer. The sign of *minus* is associated with gaining, being, as usual in astronomical work, the sign of the correction to ideal going. Externally the chronometer is of the ordinary shape, being approximately a cylinder, about 5 inches across the face and about $2\frac{3}{4}$ inches deep. It weighs about 60 oz. Lord Kelvin gives the ratio of the moment of inertia of the balance-wheel to that of the whole body as 1 : 650, but this refers to a pocket watch. For the chronometer 171 a short calculation shows that it must be of the same order.

I made two experiments, referred to as the first and second experiment. In the first the chronometer was placed in Riefler's thermostatted cell, where the beats of the two sidereal clocks, Shortt No. 0 and Shortt No 4, were audible. It was hung from a stout shelf, from the gimbal holes, by a bifilar suspension, as recommended by Lord Kelvin, but the points of the upper suspension, in place of being separated to secure difference of effect, were kept at the same distance, the depth of the chronometer below being varied by taking in links of the suspension chain, along with a screw-piece, intermediate between the links Various other refinements were introduced as they appeared necessary; for instance, a spirit-level was placed upon the face of the chronometer; leading up to the second experiment, with a much improved suspension. But it cannot be said that these refinements provided more than the exclusion of some undesirable possibilities. In consequence I have made no distinction between the various observations of the first experiment. Those of the second experiment, which will be given later, are all on the same plan.

In the first place we remark that an oscillation permitted to the cock of a pendulum, or the corresponding point of reaction to the maintenance in a chronometer, makes the timepiece go slow, which slowness increases as the point of reaction is freer, up to a certain critical position, when it changes to a large gaining rate, and a further release diminishes the gaining rate, the diminution persisting until it becomes zero. One assumes that at the critical position an infinite loss changes to an infinite gain, but I have not succeeded in determining the corresponding suspension, the nearest that I got being an unexplained change from losing to gaining, or from gaining to losing, in some cases the chronometer being carefully not wound, or disturbed in any way The rates in both cases reached a certain maximum. I should say that I saw no sign of an infinite gain or infinite loss. It may be that the rate does not pass through infinity, friction preventing it. The chronometer went irregularly in a marked way as the maxima approached.

The greatest semi-amplitude imposed on the body of the chronometer was measured at 2½ degrees, that is to say, almost double the amount shown by an ordinary clock pendulum. After a change, whether due to winding or change of suspension, it showed first a rapid increase and then a slow one, so that it was not possible to say that it had reached a maximum. It is, of course, an effect of synchronism, the greatest amplitude occurring when the taps of the maintenance agree with the natural period of the bifilar suspension.

After the experiment the chronometer returned to sensibly the same rate as it had before, some effect being shown as to whether the box was placed on a wooden shelf or on a slate shelf.

I now give the chart which represents the first experiment. I do not give the individual observations. They were very numerous, and, as will be judged from the chart, somewhat discordant, though not so much so as the chart, one rate apparently being usually held for the whole of one observation. In the chart, a single dot represents a number of observations roughly meaned. Some judgment was required, as the going at the critical point was irregular, and was not what one would expect from a good chronometer The reading of 11·44 inches—from an undetermined origin, which may be in about the right position—marks the position, presumably, of an asymptote. The clustering of many observations about this line represents an attempt to determine it more precisely.

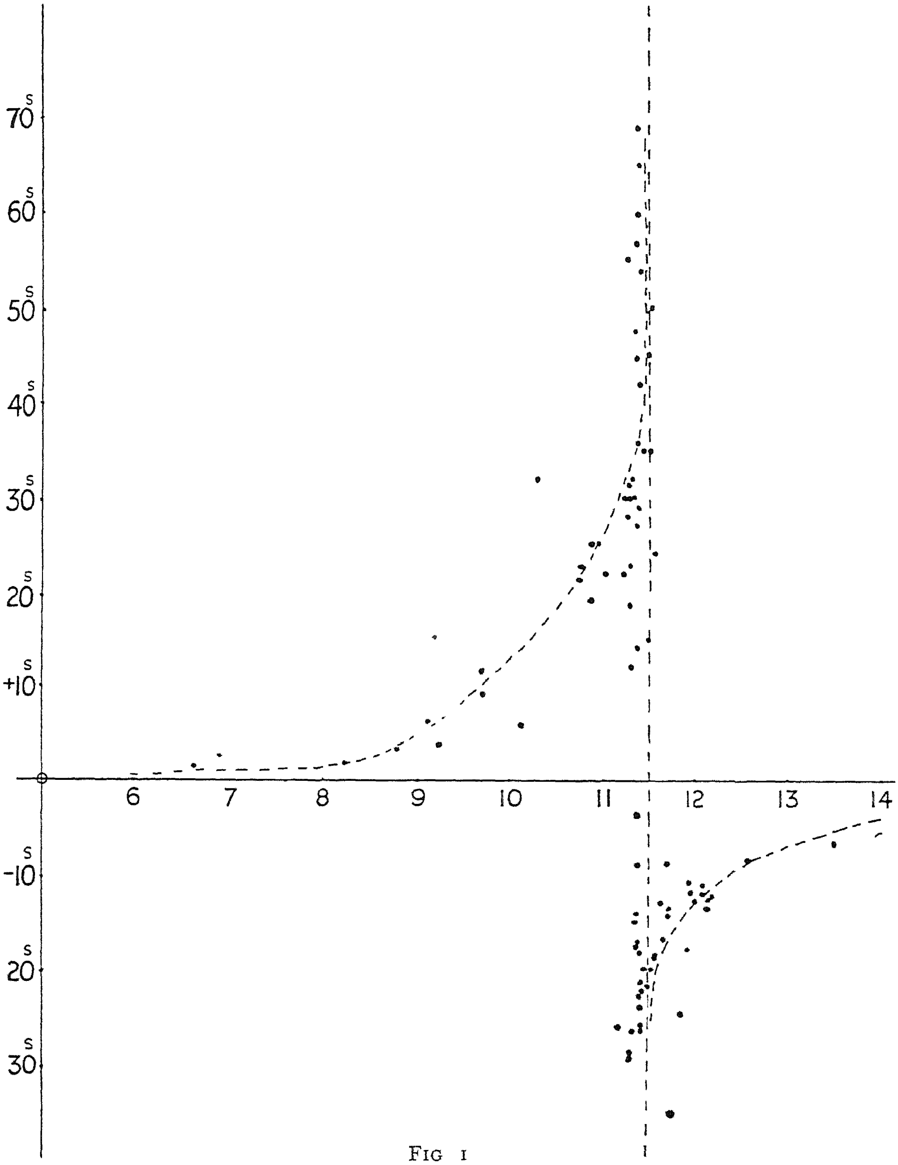

FIG 1

The "natural" rate of the chronometer is negligible compared to the amounts shown in this chart.

I shall now describe the second experiment. Thinking that the suspension might be improved, I constructed a very rigid frame of angle iron, bolted together and furnished with levelling screws It is shown in fig 2 On this a rod was placed, $\frac{3}{8}$ inch in diameter, and threaded throughout its length of $17\frac{1}{2}$ inches with a screw of 26 threads to the

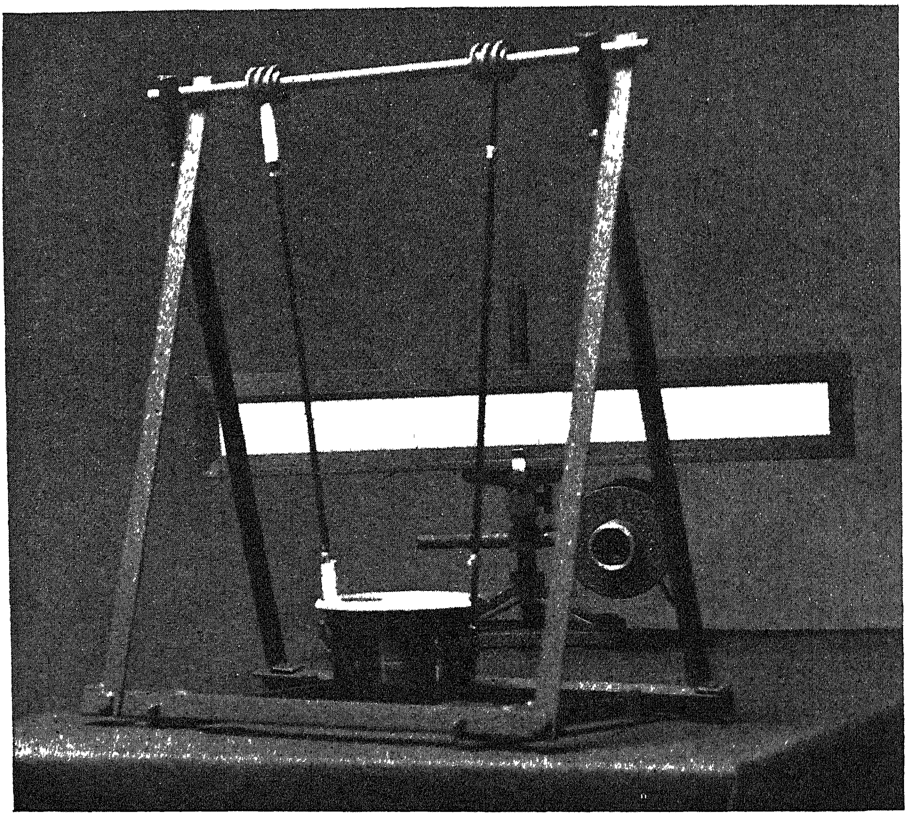

FIG. 2.

inch. Therefore one thread equals ·03846 inch, or, say, ·04 inch. The thread carried two pairs of locking-nuts which clamped securely the supports for the suspension, made to a sliding fit on the rod, permitting it to be fixed firmly at any separation. The suspension itself was made of ordinary suspension-spring, 6 mm broad and ·11 mm thick, and therefore very stiff in the direction of its breadth and very flexible in that of its thickness, besides being very strong; it was made in three parts, the direction of the first and last being perpendicular to the rod, and simply designed to take up the bending necessary for variable separation, without twisting the middle part. The lower parts were clamped to the gimbal holes and the upper parts to the supports for the suspension. In order to damp down any undesired oscillation, one pair was clothed in pith. The middle parts were in the direction of the line of the rod, and executed most of the oscillation. The dimensions were about 1½ inches, 10½ inches, 1½ inches, between the connecting blocks With these dimensions, the value of the ratio of the separation of the suspension supports to the vertical fall of the chronometer varies a little, but remains between 4 or 5, so that a single revolution of the nut changes the chronometer's position vertically by less than ·01 inch The result furnished a bifilar suspension upon a somewhat different principle to the former. The theory is repeated later, but it is obvious that when the points of suspension are united, zero effect in raising the chronometer is produced—corresponding to complete freedom of the cock of the pendulum, and that when they are separated, it tends to produce what is equivalent to a very stiff support

This particular suspension seems free from the faults that could be alleged against the other, which might make the experiment inconclusive.

There were two minor changes made in the second experiment: the apparatus was transferred to the Chemical Room, and the comparison for rate was made with the sidereal chronometer *Oram* 4645, which for ten weeks before had kept the mean rate of $0^{s}{\cdot}0$ per day, varying but little from it.

The observations are too numerous to give completely. The following statement represents them.

Corresponding to the greater freedom of the suspension, the half-oscillation of the body of the chronometer was measured, at its maximum, at more than 7½°.

As to the rate of the chronometer, I first explored the different separations, measuring them in inches with a steel rule, I found:

Separation:	12	11	10	9	8	7	6	5	4	3	2	1⅛
Rate per hour:	$+2^{s}{\cdot}2$	$+3^{s}{\cdot}0$	$+4^{s}{\cdot}6$	$+9^{s}{\cdot}3$	$-26^{s}{\cdot}2$	$-9^{s}{\cdot}6$	$-5^{s}{\cdot}4$	$-3^{s}{\cdot}3$	$-2^{s}{\cdot}7$	$-2^{s}{\cdot}2$	$-1^{s}{\cdot}9$	$-1^{s}{\cdot}8$

Concluding that the critical point lay between 9 inches and 8 inches I made a more minute examination of the region from 7 inches onwards, turning the nut two threads at a time. I found:

Separation:	7·00	7·08	7·15	7·23	7·31	7·39	7·46	7·54
Rate	$-9^{s}{\cdot}4$	$-10^{s}{\cdot}2$	$-10^{s}{\cdot}5$	$-10^{s}{\cdot}5$	$-10^{s}{\cdot}3$	$-9^{s}{\cdot}8$	$-6^{s}{\cdot}8$	$-7^{s}{\cdot}0$

Separation:	7·62	7·69	7·77	7·85	7·92	8·00	8·50	9·00
Rate:	$-6^{s}{\cdot}9$	$-7^{s}{\cdot}1$	$-6^{s}{\cdot}8$	$-7^{s}{\cdot}6$	$-7^{s}{\cdot}7$	$-7^{s}{\cdot}8$	$-10^{s}{\cdot}8$	$-25^{s}{\cdot}3$

It will be noticed that the rate, which first repeats the value obtained at the same separation before, and shows some increase with enlarged distance, as it should, then declines, and then holds on to practically the same rate, until the separation was changed by half an inch at a time; but then the value associated with a separation of 9 inches was widely different from that got before.

I then made a treble examination, at different periods, of the data about 9·0 inches, and found (repeating the last datum):

Separation		9·00	9·04	9·08
Rates	1st	$-25^{s}\cdot3$	$+13^{s}\cdot8$	$+53^{s}\cdot0$
	2nd	$+14^{s}\cdot2$	$+13^{s}\cdot6$	$+12^{s}\cdot7$
	3rd	$-16^{s}\cdot3$	$+8^{s}\cdot6$	$+7^{s}\cdot4$

I then made a more minute examination of the region from 8·50 to 8·00:

Separation:	8·50	8·42	8·38	8·35	8·31	8·27	8·24	8·20
Rate	$+31^{s}\cdot5$	$+38^{s}\cdot8$	$+21^{s}\cdot8$	$+35^{s}\cdot5$	$+22^{s}\cdot5$	$-9^{s}\cdot5$	$-14^{s}\cdot8$	$-16^{s}\cdot1$

Separation.	8·15	8·11	8·08	8·04	8·00
Rate:	$-14^{s}\cdot6$	$-15^{s}\cdot7$	$-15^{s}\cdot2$	$-13^{s}\cdot9$	$-13^{s}\cdot2$,

finally turning back to 8·50 and obtaining the rate $-18^{s}\cdot5$.

Finally I moved the clamping nuts carefully, one thread at a time, to see how long a rate of one sign could be held; I then moved them back again. The change of sign occurred after I wound the chronometer, which I was bound to do.

Separation:	9·00	8·96	8·92	8·88	8·85	8·81	8·77
Rate:	$\rightarrow +9^{s}\cdot0$	$+14^{s}\cdot2$	$+9^{s}\cdot7$	$+9^{s}\cdot6$	$+10^{s}\cdot3$	$+10^{s}\cdot8$	$+11^{s}\cdot5$

Separation:	8·73	8·69	8·65	8·61	8·58	8·54	8·50	8·46	8·42
Rate:	$\rightarrow +11^{s}\cdot3$	$+12^{s}\cdot6$	$+13^{s}\cdot6$	$+13^{s}\cdot5$	$+14^{s}\cdot4$	$+15^{s}\cdot0$	$+16^{s}\cdot7$	$+17^{s}\cdot5$	$+19^{s}\cdot3$

Separation:	8·38	8·35	8·31	8·27	8·23	8·19	8·15	8·11	8·08
Rate:	$\rightarrow +21^{s}\cdot3$	$+24^{s}\cdot3$	$+25^{s}\cdot9$	$+29^{s}\cdot1$	$+32^{s}\cdot0$	$+36^{s}\cdot1$	$+39^{s}\cdot2$	$+41^{s}\cdot7$	$+48^{s}\cdot0$
				$+39^{s}\cdot0$	$+41^{s}\cdot0$	$+43^{s}\cdot6$	$+47^{s}\cdot8$	$+52^{s}\cdot2$	$-35^{s}\cdot5 \leftarrow$

Separation:	8·04	8·00	7·96	7·92	7·88	7·85	7·81	7·77
Rate:	$\rightarrow +53^{s}\cdot8$	$+57^{s}\cdot6$	$+67^{s}\cdot2$	$+70^{s}\cdot1$	$+80^{s}\cdot0$	$+91^{s}\cdot2$	$+92^{s}\cdot8$	$+99^{s}\cdot0$
	$-32^{s}\cdot3$	$-30^{s}\cdot4$	$-27^{s}\cdot7$	$-27^{s}\cdot2$	$-25^{s}\cdot0$	$-25^{s}\cdot2$	$-24^{s}\cdot4$	$-24^{s}\cdot0 \leftarrow$

Separation.	7·73	7·69	7·65	7·62	7·58	7·54	7·50
Rate:	$+104^{s}\cdot8$	$+111^{s}\cdot8$	$+121^{s}\cdot4$	$+130^{s}\cdot8$	$+142^{s}\cdot2$	$-16^{s}\cdot1$	$-15^{s}\cdot7$
	$-22^{s}\cdot1$	$-20^{s}\cdot4$	$-19^{s}\cdot5$	$-19^{s}\cdot0$	$-17^{s}\cdot8$	$-16^{s}\cdot8$	$\leftarrow$

These appear to be reasonably consistent, so long as they belong to the same series, so that care may be taken not to introduce any disturbance. But even then, as in the second set, we get anomalies for which I know no explanation. And it seems impossible to repeat the experiment so as to get a previous result.

I shall make some remarks upon them after I have dealt with the theory, to which I now turn.

In Lamb's *Infinitesimal Calculus* (1897, p. 540) the equations of a double pendulum are given, viz.

$$x+\frac{g}{b}x(1+\mu)-\frac{g}{a}y\mu=0, \qquad x+y+\frac{g}{a}y=0,$$

where b, a are the lengths of the upper and lower suspensions, and μ the ratio of the lower weight to the upper These lead to the period-equation

$$p^4-p^2(1+\mu)g\left(\frac{1}{a}+\frac{1}{b}\right)+(1+\mu)\frac{g^2}{ab}=0,$$

where, it is to be noted, a, b are interchangeable (Lord Kelvin gives a figure (*loc cit*, p. 371) where the two suspensions are treated as different cases).

There are, of course, two periods, which are both real,

$$T_1=2\pi/p_1, \qquad T_2=2\pi/p_2$$

In ordinary cases, μ is too small to show upon a graph. I have therefore computed the result for the case $\mu=\frac{1}{10}$ derived from the formula

$$p^2\Big/\frac{g}{a}=\tfrac{1}{2}(1+\mu)\left(1+\frac{a}{b}\right)[1\pm t]$$

where

$$t^2=4/(1+\mu)\left(\frac{a}{b}+2+\frac{b}{a}\right),$$

the upper sign giving, say, $p_1{}^2$ and the lower $p_2{}^2$.

It follows that, for the case $\mu=0$, when the period equation becomes $\left(p^2-\frac{g}{a}\right)\left(p^2-\frac{g}{b}\right)=0$ the solution $p^2=\frac{g}{a}$ which represents the undisturbed period, coincides with $p_2{}^2$ when $b<a$, and with $p_1{}^2$ when $b>a$, the remaining portions being parts of a rectangular hyperbola, with a sharp angle occurring where they cross.

The least difference between $p_1{}^2$ and $p_2{}^2$ is found where

$$b/a=(1+\mu)/(1-\mu),$$

when

$$p_1{}^2=\frac{g}{a}(1+\mu^{\frac{1}{2}}), \qquad p_2{}^2=\frac{g}{a}(1-\mu^{\frac{1}{2}}).$$

The results are shown in fig. 3.

If we put $p^2=\frac{g}{a}(1+z)$, then z is the ordinate of fig. 1, p. 15, taking due account of the difference of units, and also the convention of signs. If we apply Lamb's period equation, we have

$$p_1{}^2+p_2{}^2=(1+\mu)\frac{g}{a}\left(1+\frac{a}{b}\right), \qquad p_1{}^2p_2{}^2=(1+\mu)\frac{g^2}{a^2}\cdot\frac{a}{b};$$

therefore, writing

$$p_1^2 = \frac{g}{a}(1 + z_1), \qquad p_2^2 = \frac{g}{a}(1 + z_2)$$

and

$$p_0^2 = \frac{g}{a}(1 + z_0)$$

where z_1, z_2 are of opposite signs and z_0 is usually zero; eliminating the ratio a/b, we find

$$\mu = \frac{1 + z_1 + z_2}{1 + z_0} - \frac{(1 + z_1)(1 + z_2)}{(1 + z_0)^2},$$

or if $z_0 = 0$,

$$\mu = -z_1 z_2.$$

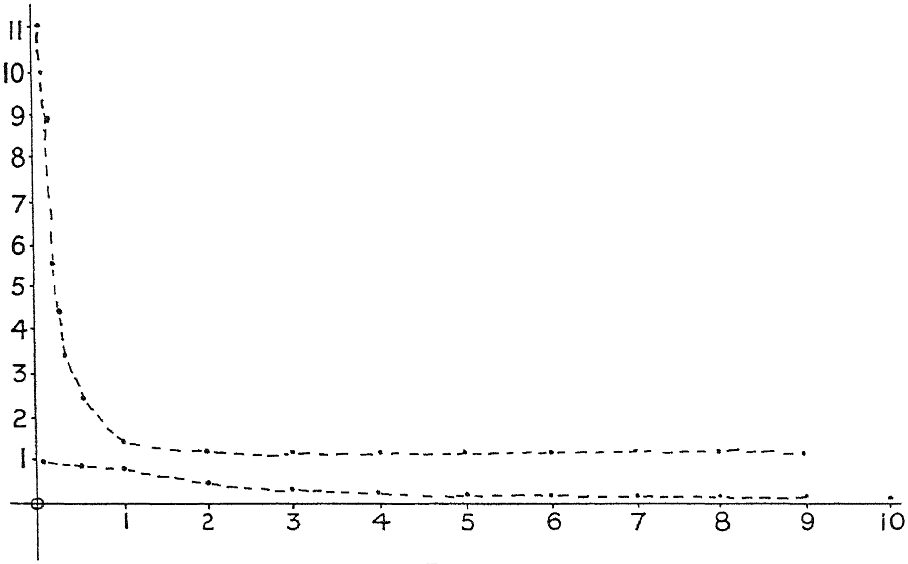

FIG. 3.

Putting the largest values found above, viz

$$z_1 = +22/3600, \qquad z_2 = -66/3600,$$

this gives

$$\mu = +\cdot000112 = 1/8900,$$

which is certainly too small. In fact the application of the double pendulum is instructive and simple, but it is in some respects misleading, because an actual clock requires maintenance against friction In all clocks some friction is present, though attempts are made to reduce it to the smallest amount. The provision of a maintenance alters the character of the motion completely. The study of a maintenance was included in the first paper of this series (*Proc. Roy. Soc. Edin.*, vol xxxviii, 1918, p. 97). I thought when I wrote it that this was the first consideration, but I found afterwards that very much the same point was included by Rayleigh, *Theory of Sound*, vol. i (2nd ed , § 68*a*, p. 79). I shall, however, quote from my own discussion, which is more to the point that I want to make. The essential point is that the period of the maintenance is strictly equal to the period of motion A motion supplied by a maintenance is a species of forced vibration, and, of course, a forced vibration may be of any period. But it appears that the maintenance may coincide strictly with the period of the oscillation without producing an infinite excursion; for if the equation is

$$x+\kappa\dot{x}+n^2x=\mathrm{A}\cos\tau+\mathrm{B}\sin\tau \qquad [\tau=n't+\epsilon]$$

we have

$$\frac{n'}{n}=\left(1+\frac{\mathrm{B}^2}{\mathrm{A}^2}\frac{\kappa^2}{n^2}\right)^{\frac{1}{2}}-\frac{1}{2}\frac{\mathrm{B}}{\mathrm{A}}\frac{\kappa}{n},$$

and an exact solution

$$x=\frac{\mathrm{A}}{\kappa n'}\sin\tau$$

The finite values of κ, A, B separate n' from n. The presence of the first power of κ in n' is noteworthy

In the case of a suspended chronometer, the maintenance is due to the winding of the chronometer. The motion dies away very quickly when the chronometer runs down.

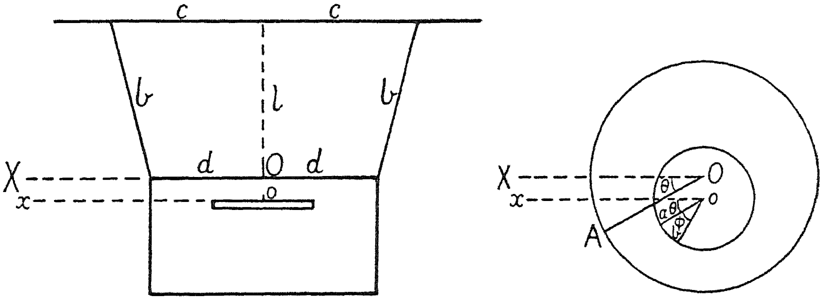

FIG. 4

We may write down the equations that express the motion. Without unnecessary complications, they are

$$\mathrm{MK}^2\theta+\mathrm{P}\dot{\theta}+(\mathrm{M}+m)g\mathrm{L}\theta=-\mathrm{S} \quad . \quad . \quad . \qquad (1)$$

$$mk^2(\theta+\phi)+\mathrm{Q}\dot{\phi}+\mathrm{R}\phi=\mathrm{S} \quad . \quad . \quad . \qquad (2)$$

where M, m	are the masses of the body and balance wheel respectively,
MK^2, mk^2	their moments of inertia,
S	the couple due to winding,
P	the frictional parameter in the suspension,
Q	the same in the chronometer,
R	is provided by the balance-spring.

In the case of a bifilar suspension, it may be shown that, in the neighbourhood of the point of rest,

$$L = cd/l,$$

where c is the half separation of the upper points,
d ,, ,, ,, lower ,,
l the depth of the centroid of M, m below the fixed bar.

It is the endeavour of chronometer-makers to render S constant and Q zero The latter is done by polishing the pivots until they are "black," making them small, especially the quick-moving ones, housing them in jewels, oiling them, and so forth. Nevertheless Q is quite appreciable. Much more so is P—which has not received any of these precautions—at least in the first experiment. As to S, as the spring runs down and exerts less force, its reduction is compensated by putting in the fusee, so that S may consist of a series of constant taps If we write

$$S = \alpha_1 \cos \tau + \beta_1 \sin \tau + \alpha_2 \cos 2\tau + \beta_2 \sin 2\tau + \ldots$$

this is the most general formula of period (τ); by a proper choice of the coefficients, it can be made to represent any sequence of sharp taps of that period, the action lying dormant between. The first two terms represent a sinusoid with its gradual growth and are evidently not constant. Still, for simplicity, I shall omit the later terms and suppose that the first two represent S Evidently there is no question of principle involved It is evident, too, that it is no assumption to write

$$S = \nu\phi + \pi\dot{\phi} + \rho\theta + \sigma\dot{\theta}, \quad . \quad . \quad . \quad (3)$$

thereby introducing four parameters, ν, π, ρ, σ—which may or may not be zero—to represent the relations of S, the winding couple, with the reactions corresponding to the two co-ordinates. We may expect ν, π, ρ, σ to be not wholly independent. Also, they necessarily involve the relation of the winding with the motion it induces in the co-ordinates

We remark that the chief function of the winding couple in a fixed chronometer is to annul Q.

There is a solution of the equation (1), (2), of the nature of a forced vibration, if the same period runs through all the terms. Put then

$$\theta = A \cos pt + B \sin pt, \qquad \phi = C \cos pt + D \sin pt, \quad . \quad . \quad (4)$$

where A, B, C, D are *not* arbitrary, but must satisfy equations (1), (2), which are specified in terms of the winding couple.

Equating to zero the coefficients of $\cos pt$ and $\sin pt$ in each equation, we get

$$
\begin{aligned}
0 &= A\{-MK^2p^2+(M+m)gL+\rho\}+B(P+\sigma)p+C\nu+D\pi p,\\
0 &= -A(P+\sigma)p+B\{-MK^2p^2+(M+m)gL+\rho\}-C\pi p+D\nu,\\
0 &= A\{-mk^2p^2-\rho\}+B\sigma p+C\{-mk^2p^2+R-\nu\}+D(-Q+\pi)p,\\
0 &= -A\sigma p+B\{-mk^2p^2-\rho\}+C(Q-\pi)p+D\{-mk^2p^2+R-\nu\}.
\end{aligned}
$$

Eliminating A : B : C : D, we get an equation which appears to be of the 8th degree in p But the determinant is of the form

$$
\begin{vmatrix}
\alpha_1, & \beta_1, & \gamma_1, & \delta_1\\
-\beta_1, & \alpha_1, & -\delta_1, & \gamma_1\\
\alpha_2, & \beta_2, & \gamma_2, & \delta_2\\
-\beta_2, & \alpha_2, & -\delta_2, & \gamma_2
\end{vmatrix}
$$

and this may be shown to be equal to

$$(\alpha_1\gamma_2-\beta_1\delta_2-\gamma_1\alpha_2+\delta_1\beta_2)^2+(\alpha_1\delta_2+\beta_1\gamma_2-\gamma_1\beta_2-\delta_1\alpha_2)^2,$$

wherefore, since imaginary results have no significance, the conclusion is that one result leads to two equations, viz :

$$
\begin{aligned}
0 &= \{-MK^2p^2+(M+m)gL+\rho\}\{-mk^2p^2+R-\nu\}+p^2(P+\sigma)(Q-\pi) \qquad (5)\\
&\quad +\nu(mk^2p^2+\rho)+p^2\pi\sigma,\\
0 &= \{-MK^2p^2+(M+m)gL+\rho\}(-Q+\pi)+(-mk^2p^2+R-\nu)(P+\sigma) \qquad (6)\\
&\quad -\nu\sigma+(mk^2p^2+\rho)\pi,
\end{aligned}
$$

where the second has been divided by p.

The former may be considered the period equation, and the latter a condition imposed upon the coefficients ν, π, ρ, σ Consider the former Replace L, R, ν by three constants a', b', μ', this is legitimate provided it does not imply any necessary relations between them; write then

$$
\begin{aligned}
R-\nu &= mk^2\frac{g}{a'},\\
-\nu &= MK^2\mu'\frac{g}{a'}, \qquad . \quad . \quad . \quad . \quad (7)\\
(M+m)gL+\rho &= MK^2(1+\mu')\frac{g}{b'},
\end{aligned}
$$

where we remark that if we make $\mu'=mk^2/MK^2$, it implies $R=0$; therefore this value, which is the natural one to take, is forbidden. As already said, the equations differ essentially from those of the double pendulum owing to the presence of R, which is due to the timing spring.

Dividing equation (5) by $MK^2 . mk^2$, it may then be put

$$
\begin{aligned}
0 &= p^4-p^2\left\{(1+\mu')g\left(\frac{1}{a'}+\frac{1}{b'}\right)-\frac{PQ-P\pi+Q\sigma}{MK^2 . mk^2}\right\} \quad . \quad . \quad (8)\\
&\quad +(1+\mu')\frac{g^2}{a'b'}-\mu'\frac{g}{a'}\frac{\rho}{mk^2}.
\end{aligned}
$$

This may be identified with Lamb's period equation (Lamb's equation contains three adjustable parameters, which is more than sufficient to identify it with any quadratic equation)

$$0 = p^4 - p^2(1+\mu)g\left(\frac{1}{a}+\frac{1}{b}\right) + (1+\mu)\frac{g^2}{ab},$$

if we write

$$a' = a + \delta a, \qquad b' = b + \delta b, \qquad \mu' = \mu + \delta\mu,$$

but I have not succeeded in reducing the substitution to any significant form.

A point of some theoretical interest is the following. The period equation is presented without any reason to prefer one solution to the other. But in the actual experiment, the chronometer selects one solution first—viz. losing—and afterwards changes over to the other.

Lord Kelvin remarked (*loc cit.*, p. 382), in the critical position, "the chronometer may be compelled to take up either mode by properly timed touches with the finger, and it continues vibrating accordingly when left to itself." But this only refers to the case when the roots are nearly equal, and then, I suppose, friction may enter to turn the scale. In the other cases, the diagram of fig. 3 may be consulted with advantage It will be seen that the actual period selects the nearer of the two possible periods. I draw an analogy between the two trajectories possible to a projectile, when the positions of departure and arrival are given and also the field of force. One corresponds to minimum action and one to maximum action. I presume this is the cause for the selection of the period by the chronometer.

From the practical point of view a more significant remark may be made as follows. It is a known plan to bring a pendulum on time by disturbing it. In particular it has been used with marked success in the case of the Shortt clocks, in the control of the slave by the master. There the operation is always in the sense of making the pendulum of the slave go a little faster, by interposing a hit-or-miss device, which, when it operates, holds it up over its greatest excursion. But it may be used to make the pendulum go slower, the rule being that a change which assists gravity (in the case of a pendulum) makes the clock go faster, and a change in the contrary direction makes it go slower. Also, it is to be noted that identically the same disturbance operates in a different way according to its point of application. If applied near the place of greatest excursion, it changes chiefly the phase or time-keeping; but it changes chiefly the arc if applied near the point of greatest motion. Now the essential feature of a suspended chronometer is just a disturbed pendulum We try to make the disturbance regular, but we have no control over the point of application. Besides, there is an accidental element in all controls, varying only in degree I conclude then that the operation is to be expected to be uncertain, and very uncertain at the critical point, even to the extent of changing over from gaining to losing, at a suspension which it is impossible to fix precisely.

An indirect bearing upon clock practice may be mentioned. At Edinburgh Observatory we use the sidereal clock Shortt No 4 to mark the seconds on the Wetzer chronograph. This is done by the pendulum of the slave, the end of each half-minute being marked, as usual, by the slave re-winding itself, and then the master, which the slave releases, recording its control of the slave. It is found that the one-second signals agree closely with those of the slave winding itself, but that there is a sensible uncertainty in the control The microchronograph confirms this. One finds about as many differing by $\pm {}^{s}\cdot 01$ as are found at the mean, with, of course, some outlying cases Besides, the means, taken over twenty days, drift about by perhaps half that amount. Possibly the essential uncertainty of the operation of the control may be the cause, which, needless to say, requires to be taken into account.

Having made the experiment, I considered whether it had any bearing upon the practice of clock- and chronometer-makers. I have found only minor and indirect ones. Irregular going when the oscillating piece is disturbed would, it seems, forbid any other. Lord Kelvin speaks of it as a method of changing the hands of a chronometer without injury. But I do not see that there is any advantage over the known practice, viz. when the hands are slow, to move them forward by sudden jerks at the central square, choosing the moments when the movement is at rest; and when the hands are fast, to stop the balance with a peg, starting it again by rotation of the body at the right moment. An easy method is desirable for changing the error by any fraction of a second, as well as a whole number of them, and it may be done with a suspended chronometer, provided we have a correct pattern, where we can listen to the coincidences; but otherwise it is unsafe. Any change of rate is not to be recommended.

(*Issued separately May* 19, 1936.)

IV.—Some Experiments having Particular Reference to the Flow of Water along Short Capillary Tubes connecting Two Vessels with Free Surfaces.

By **J. Allen,** M.Sc., Assoc.M.Inst.C.E., University of Manchester. *Communicated by* Professor MONTAGU F DRUMMOND, M.A. (With Five Text-figures.)

(MS received January 3, 1936 Read March 2, 1936)

1. LIST OF SYMBOLS.

$\bar{v}$ = mean velocity, $=\frac{Q}{a}$.
Q = discharge per second.
d = bore of tube
a = cross-sectional area $=\frac{\pi d^2}{4}$.
l = overall length of tube
μ = coefficient of viscosity in absolute units
ν = coefficient of kinematic viscosity $=\frac{\mu}{\rho}$.
ρ = density of water.
h = head = vertical distance between water-surfaces in reservoirs
m = hydraulic mean depth or radius $=\frac{d}{4}$.
f = a friction coefficient.
R = "Reynolds' Number" $=\frac{\bar{v}d}{\nu}$.

2 OBJECT OF RESEARCH

The work described in the present paper was designed to investigate the losses of energy accompanying the flow of water from one vessel or reservoir to another through capillary tubes, more especially those having a comparatively small ratio of length to bore It is believed that the experiments cover a range of conditions not previously explored.

3. GENERAL DISCUSSION OF THE PROBLEM.

The equation of head-balance in the discharge of fluid through such tubes may be expressed in the form:

h = loss at entry + loss due to general motion along the tube + loss at exit . (1)

Of these losses, that at entry may be assumed to equal $C_1\frac{\bar{v}^2}{2g}$, and C_1 would tend to become zero in the ideal case of streamline motion through a perfect bell-mouth. It is, however, generally considered by hydraulic engineers that C_1 for the comparatively large commercial pipes used in practice is approximately 0·05 when a suitable bell-mouth is provided. With a sharp-edged inlet, the loss would be expected to be greater, owing to the deformation of the streamlines of the fluid and the contraction and subsequent re-enlargement of the stream

The loss due to viscous resistance in streamline motion along a cylindrical tube is, ideally,

$$\frac{32\nu l\bar{v}}{gd^2}, \qquad . \quad . \quad (2)$$

while the head equivalent to kinetic energy of viscous motion is

$$2\left(\frac{\bar{v}^2}{2g}\right). \qquad . \quad . \quad . \quad . \quad (3)$$

It has long been realised, however, that equations (2) and (3) are not applicable to every portion of the tube, since as the fluid enters the mouth the velocity distribution over the cross-section of the tube is not parabolic· it only assumes the normal parabolic form indicated by analysis after passing through a transition stage.

In short, it is well known that if the capillary tube method is used to determine the viscosity of liquid, it is not strictly accurate to calculate ν from the observed head and discharge by the simple formula

$$h=\frac{32\nu l\bar{v}}{gd^2}. \qquad . \quad . \quad . \quad . \quad (4)$$

For information as to the devices which have been suggested to correct the simple formula, *e.g.* increasing the length, l, of the tube by an amount proportional to the radius, and the substitution of a quantity $\frac{k\bar{v}^2}{2g}$ in place of the ideal kinetic head $\frac{2\bar{v}^2}{2g}$, the reader is referred to the extensive literature on the subject, and in particular to the publications named at the end of this paper (Barr, 1931; Dryden, Murnaghan, Bateman, 1932; Gibson, 1913).

As will appear later, in the discussion of results, the present experimental observations are reasonably represented by an equation

$$h=\frac{A\nu l\bar{v}}{gd^2}+\frac{B\bar{v}^2}{2g}, \qquad . \quad . \quad . \quad . \quad (5)$$

wherein A and B are each functions of $\frac{l}{d}$.

4. APPARATUS AND EXPERIMENTAL PROCEDURE.

Fig. 1 illustrates the general arrangement of the apparatus. The capillary tube was supported in rubber stoppers plugged into brass tubes projecting from the sides of two copper reservoirs, each of which was 10 in long, 6¼ in. wide and 10 in. deep The centre of the capillary tube was situated at a height of 0·7 in. above the bottom of the reservoirs, and the supply of water to the first reservoir was obtained from a gravity tank, provided with a ball-valve. This gravity tank (not shown in fig. 1) was connected, by means of a pipe containing a finely adjustable needle-valve, to a vertical cylinder equipped with a calibrated orifice from which a stream fell upon a chute and was guided gently into the end of the copper box remote from the capillary tube. The object of the calibrated orifice was merely to indicate, for convenience, the approximate rate of discharge at any instant.

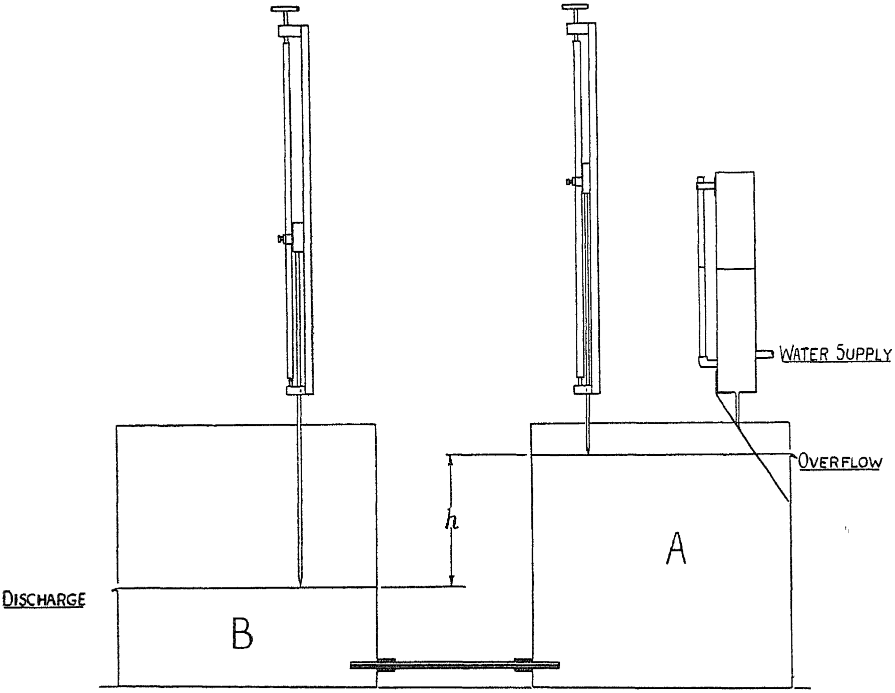

FIG 1

In order to minimise the time taken for the levels in the two boxes to become steady, a hole ½ in. in diameter was drilled at a height of some 8·8 in. from the bottom of the first reservoir, A, in the side remote from the capillary tube, while the second reservoir, B, was provided with seven such holes in the side opposite the entering tube. These holes could readily be plugged with corks when not required and enabled heads of approximately 16, 12, 8, 7, 6, 5 and 4 cm. respectively to be maintained by the procedure of adjusting the valve in the pipe supplying reservoir A so that a surplus trickle was always passing through the hole in its side. The second reservoir, B, was initially filled with water to the level of its side-holes, and the discharge from that hole collected when conditions had become steady. For the purpose of measuring this discharge, the time to collect known, carefully measured volumes was observed by means of a stop-watch graduated in tenths of a second and checked day by day with reference to (i) an electric clock, (ii) a standard watch.

Temperatures were observed by means of a centigrade thermometer marked in tenths of a degree and calibrated against a standard thermometer.

The difference of water-levels in the two reservoirs was determined by means of pointer-gauges whose relative zero readings were established from time to time by means of static water-levels These pointer-gauges were equipped with verniers readable to ±0·0025 in (or ±0·00635 cm.). A number of tests were also made, as a check, using a vertical differential toluene-water gauge.

The apparatus was mounted on a light steel framework with suitable stiffening members, and so arranged that the copper reservoirs could slide between guides to facilitate the use of capillary tubes of different lengths.

For the measurement of the bore, or inside diameter, of the tubes the "mercury pellet" method was adopted, a thread of mercury being measured at various positions in the capillary by means of a travelling microscope, the thread being afterwards weighed. Microscopic examination of the ends of the tubes, which were ground normal to the axis, revealed a general tendency to slight ellipticity in the hole, but this was, in all probability, accentuated at the ends due to the action of grinding.

The outside diameters of the tubes were measured at a number of sections by means of a micrometer screw-gauge.

5. ASSUMED VALUES OF VISCOSITY AND DENSITY OF WATER.

The values assumed for viscosity and density were extracted from the *International Critical Tables* (1929).

6. VARIATION OF TEMPERATURE DURING TESTS.

No attempt was made to control the water temperature, which varied from time to time between 8·57° and 15·24° C., but, in any one test, rarely altered by more than 0·1° C.

7. EXPERIMENTS PERFORMED

Tests were made under conditions summarised in the following table:—

Expt. No.	Tube.	Length, cm	Bore, cm.	Outside Dia., cm	$\frac{l}{d}$.	Conditions.
1	IA	4·23	0·1464	0·229	28·8(7)	Projecting 0·40 cm. in each reservoir
2	IB	6·30	0·1434	0·228	43·9(3)	Projecting 0·50 cm. in each reservoir
2A	IB$_1$		. .	.		Flush in A, projecting in B.
2B	IB$_2$				. .	Flush in B, projecting in A.
3	II	6·30	0·1295	0·559	48·6(0)	Projecting 0·50 cm. in each reservoir
4	IIIA	8·05	0·2450	0·760	32·8(5)	Projecting 0·89 cm. in each reservoir
5	IIIB	16·00	0·2453	0·761	65·22	Projecting 2·82 cm. in each reservoir
5A	IIIB$_1$	.	. .			Do., but bell-mouth entry.
6	IIIC	27·00	0·2509	0·765	107·6	Projecting 2·82 cm. in each reservoir
6A	IIIC$_1$	. .	. .			Do., but bell-mouth entry.
6B	IIIC$_2$	.	.	.		Do., but wider bell-mouth entry.
6C	IIIC$_3$	.	.	.		Sharp inlet, bell-mouth exit.
7	IIID	38·05	0·2451	0·788	155·2	Projecting 2·82 cm. in each reservoir.
7A	IIID$_1$		. .	. .	.	Projecting 2·82 cm. in each reservoir, but bell-mouth entry.
8	IIID^1	37·60	0·2451	0·788	153·4	Projecting 2·82 cm. in each reservoir

In all the experiments, the water surface in reservoir A was approximately 21·0 cm. above the axis of the capillary tube, the various heads being obtained by altering the level in reservoir B.

8. RESULTS.

(*a*) The general field of the main results is demonstrated in fig. 2, by the plotting of $\frac{2gh}{\bar{v}^2}\cdot\frac{d}{4l}$ against $\frac{\bar{v}d}{\nu}$ logarithmically. In explanation of this method of plotting, suppose we take

$$h=\frac{fl\bar{v}^2}{2gm} \quad . \quad . \quad . \quad . \tag{6}$$

so that

$$f=\frac{2gmh}{l\bar{v}^2}=\frac{2gh}{\bar{v}^2}\cdot\frac{d}{4l}. \quad . \quad . \quad . \quad . \tag{6A}$$

If the tube were very long and the losses of energy at inlet and exit negligible in comparison with the loss in the main motion,

$$h \text{ would} = \frac{32\nu l\bar{v}}{gd^2}, \quad . \qquad . \qquad (7)$$

and, from equations (6) and (7),

$$f \text{ would then} = \frac{16}{\frac{\bar{v}d}{\nu}} = \frac{16}{\mathrm{R}}. \qquad . \qquad (8)$$

The line representing equation (8) is drawn in fig. 2; the plotted points from the experimental tubes depart more and more from it as $\frac{\bar{v}d}{\nu}$ increases and as $\frac{l}{d}$ decreases.

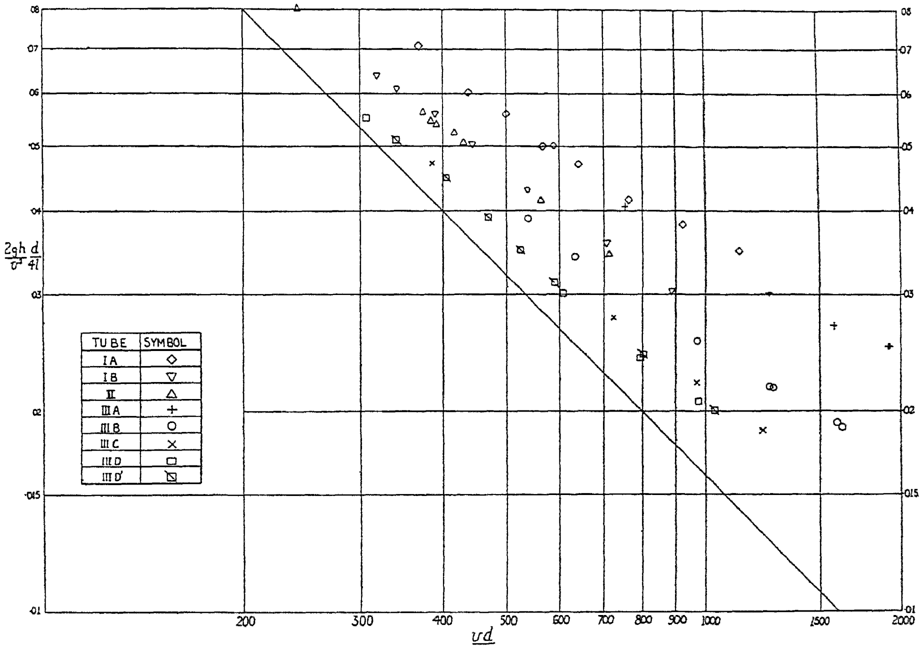

FIG. 2.

Fig. 2 also serves to show the wide range of values of Reynolds' number covered by the tests; it will be noticed that in the experiments with tube IIIA the lower critical number $\left(\frac{\bar{v}d}{\nu} = \text{approximately } 2000\right)$ was approached.

(*b*) If now, the quantity $\frac{2gh}{\bar{v}^2}$ is plotted against $\frac{l}{d}\cdot\frac{1}{\mathrm{R}}$, it is found impossible to regard the points as belonging to one general curve which is applicable to all the tubes.

(*c*) On plotting $\frac{2gh}{\bar{v}^2}$ against $\frac{1}{\mathrm{R}}$ for the various tubes, however, it is found that, within the accuracy of the experiments, a series of straight lines is obtained Such plotting has been carried out by the author to a very large scale, a small scale reproduction of his diagram is shown in fig 3

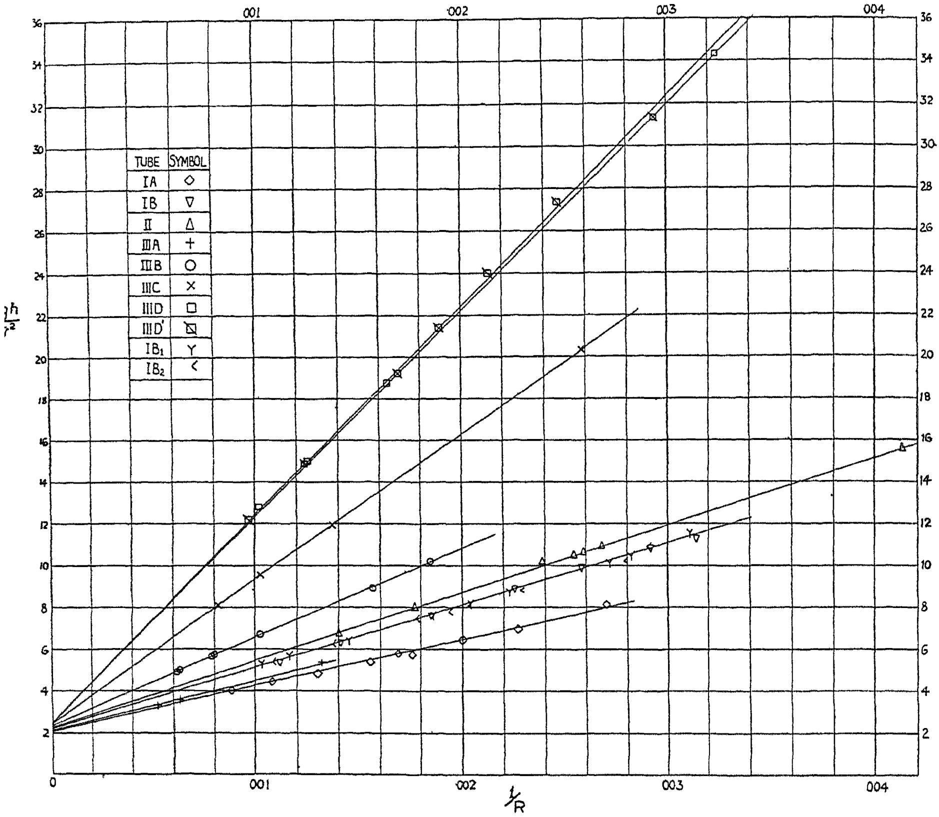

FIG. 3.

It appears, then, that $\frac{2gh}{\bar{v}^2} = \alpha + \beta\frac{l}{d}\cdot\frac{1}{R}$, the values of α and β varying from tube to tube but being sensibly constant for any one tube over the respective ranges of R covered by the tests.

By a process of trial and error, it has been possible to obtain the following empirical expressions for α and β, which give a close general agreement with the plotted points for all the tubes:

$$\alpha = 2{\cdot}60 - 15{\cdot}5\frac{d}{l} \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

$$\beta = 64{\cdot}00 + 1{\cdot}33 \times 10^6\left(\frac{d}{l}\right)^{3{\cdot}44} \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

These equations give the following values of α and β for the various tubes:—

Tube.	$\frac{l}{d}$.	α.	β.
IA	28·8(7)	2·06	76·58
IB	43·9(3)	2·25	66·97
II	48·6(0)	2·28	66·10
IIIA	32·8(5)	2·13	72·07
IIIB	65·22	2·36	64·76
IIIC	107·6	2·46	64·14
IIID	155·2	2·50	64·04
IIID^1	153·4	2·50	64·04

The corresponding lines are drawn in fig. 3 and, having regard to the considerable effect of small errors in measurement of the bore of the tubes, show definitely a close agreement with the general results of the tests.

If now, instead of the equation

$$\frac{2gh}{\bar{v}^2}=\alpha+\beta\frac{l}{d}\cdot\frac{1}{R}$$

we suppose

$$\frac{2gh}{\bar{v}^2}=\alpha+64\cdot00\frac{l+nd}{d}\cdot\frac{1}{R},$$

the following are the applicable values of n:—

Tube.	$\frac{l}{d}$.	n.
IA	28·8(7)	5·68(9)
IB	43·9(3)	2·06(5)
II	48·6(0)	1·60(3)
IIIA	32·8(5)	4·14(0)
IIIB	65·22	·913
IIIC	107·6	·215
IIID	155·2	·064
IIID^1	153·4	·064

Further, in a very long tube $\left(\frac{l}{d}\rightarrow\infty\right)$, the transition length experienced before true parabolic distribution of velocity is attained will be comparatively negligible; the value of β given by equation (10) is then appropriately 64·00. Assuming also that there is a loss due to sudden contraction and re-expansion at entrance of the order of $0\cdot6\,\frac{\bar{v}^2}{2g}$ and that the whole of the kinetic head $\frac{2\bar{v}^2}{2g}$ is rejected at exit, α is then appropriately evaluated as 2·60 by equation (9). It is not to be considered, however, that the various "losses" are truly separable by terms such as α and β, but rather that these terms, by absorbing together the effects of the phenomena involved, render the ideal equation empirically applicable to observed fact.

Extrapolation to the case of an orifice $\left(\frac{l}{d} \to 0\right)$ is impracticable, but the small values of α associated with the shortest tubes are doubtless attributable to the fact that, in those cases, the parabolic distribution of velocity was never attained and consequently the head rejected at exit never reached the magnitude of $\frac{2\bar{v}^2}{2g}$.

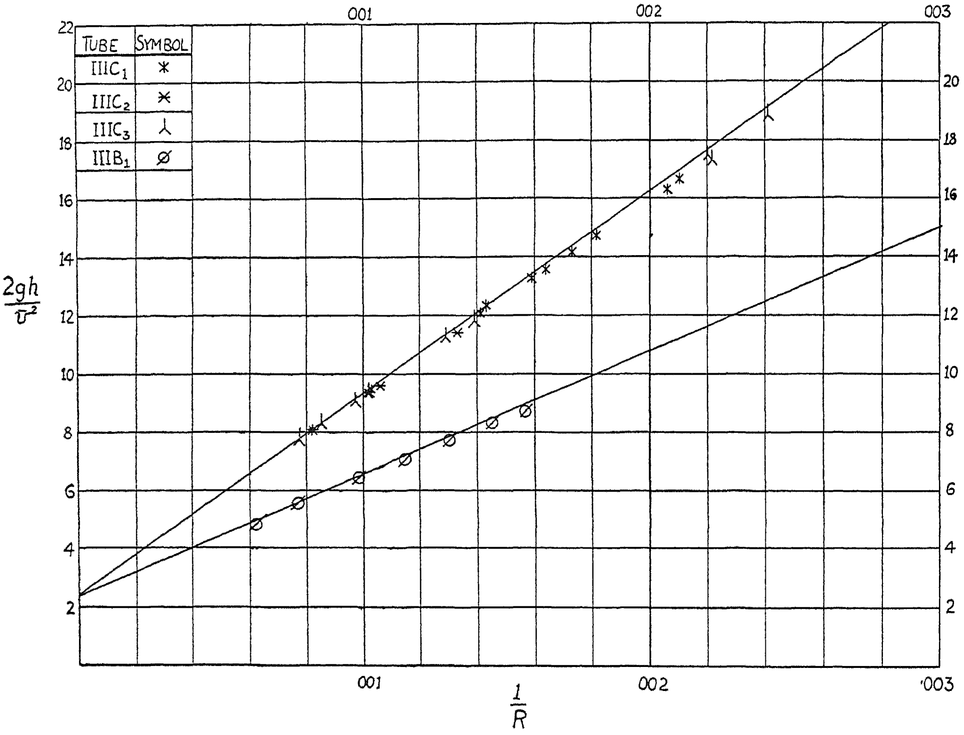

FIG. 4.

It is also possible, if not indeed probable, that the empirical rules deduced would be inapplicable to the same tubes if tested at values of Reynolds' number outside the range of the present experiments. They would, of course, be invalid at numbers closer to and beyond the breakdown of streamline into turbulent motion; it is possible also that they would fail to apply satisfactorily at lower Reynolds' numbers than dealt with and exhibited in fig. 2.

(*d*) The general expressions derived apply, within the limits of the experiments, to all the tubes independently of their ratio of outside to inside diameters, although that ratio varied between limits of 1·59 and 4·31.

(*e*) The results obtained with tube IB when tried (i) flush with the wall of reservoir A and projecting in B, and (ii) projecting in A but flush in B, differed but slightly and irregularly from those obtained with the same tube having its ends projecting in both reservoirs. In effect no difference was discernible.

(*f*) By grinding the end of the tube to a bell-mouth shape, it was possible to carry out the tests numbered 5A, 6A, 6B, 6C and 7A in the table of section 7.

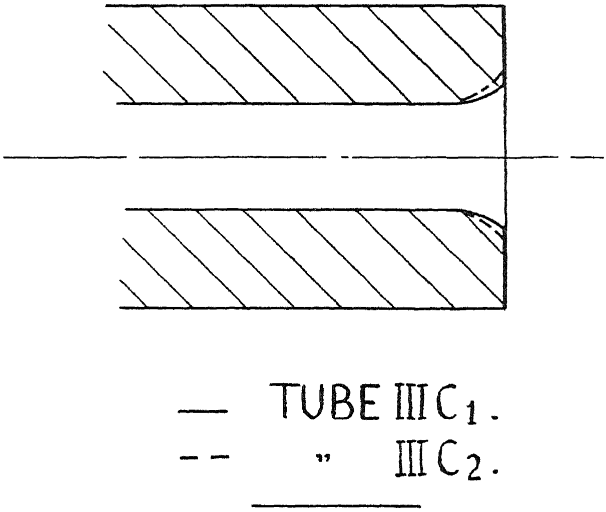

Fig. 5—General Shape of Bell-mouthed End.

Considering first the experiments in which this bell-mouthed end was used as the inlet, it was impossible to discern any effect with the longest tube IIID$_1$. With the shorter tubes, IIIC$_1$, IIIC$_2$ and IIIB$_1$, however, a measurable difference is seen on plotting, as in fig. 4, $\frac{2gh}{v^2}$ against $\frac{1}{R}$. The lines drawn in fig. 4 are transferred from fig. 3, which relates to the tests with sharp ends.

Fig. 4 also shows the effect of a bell-mouth exit in the case of tube IIIC$_3$. Here again there is a slight gain from the more gradual increase of area at the exit end, enabling a portion of the kinetic head to be converted into pressure.

The experiments with tube IIIB$_1$ confirm the reduction of head consequent upon grinding the inlet end of tube IIIB to form a bell-mouth.

Actually there is a marked tendency for the effect of the bell-mouth entrance to be more pronounced with the smaller Reynolds' numbers.

The general shape of the bell-mouth as ground in these tubes is indicated in fig. 5. The grinding was facilitated by first machining a brass template to act as a guide in producing the shape required.

9. Assuming the equation

$$\frac{2gh}{\bar{v}^2}=\alpha+\beta\frac{l}{d}\cdot\frac{1}{R},$$

where α and β have the values, for the sharp-ended tubes, given by equations (9) and (10), we obtain

$$\nu=\left\{\frac{2gh}{\bar{v}^2}-\alpha\right\}\frac{\bar{v}d^2}{\beta l}. \qquad (11)$$

The corresponding "ideal" equation is

$$\nu=\left\{\frac{2gh}{\bar{v}^2}-2\right\}\frac{\bar{v}d^2}{64l}. \qquad (12)$$

Accordingly, the ratio of the true value of ν to that which would be derived from a given set of observations by using the simple equation in which $\alpha=2{\cdot}00$ and $\beta=64{\cdot}00$ is

$$\left(\frac{\dfrac{2gh}{\bar{v}^2}-\alpha}{\dfrac{2gh}{\bar{v}^2}-2}\right)\cdot\frac{64}{\beta}. \qquad (13)$$

The values given by this expression, for some of the tubes, are set out in the table below, corresponding to various values of $\frac{2gh}{\bar{v}^2}$.

Tube.	$\frac{l}{d}$	$\frac{2gh}{\bar{v}^2}$.								
		4.	5	6	8.	10.	12	14.	20	30
IA	28 8(7)	·811	..	823	·827	..	..	..	.	..
II	48 6(0)	.	.	.	923	·934	·941	·946	.	
IIIB	65 22	..	·868	·894	·927	942	..	.	.	..
IIIC	107·6	..	..	.	·922	·941	·952	960	·972	.
IIID	155 2	.	.	..	..	·937	949	·957	·971	·981

10. In plotting the diagrams (figs 2–4), observations under closely coincident conditions have, in certain few cases, been averaged to give the points shown.

11. SUMMARY

The present investigation of the discharge of water from one vessel or reservoir to another with free surfaces, through capillary tubes of various ratios of outside to inside diameters between 1·59 and 4·31 and of various $\frac{l}{d}$ ratios between 28·8(7) and 155·2, indicates that the behaviour as a whole is represented by

$$\frac{2gh}{\bar{v}^2} = \alpha + \beta\frac{l}{d}\cdot\frac{1}{\text{R}},$$

where

$$\begin{cases} \alpha = 2{\cdot}60 - 15{\cdot}5\dfrac{d}{l}, \\ \beta = 64{\cdot}00 + 1{\cdot}33 \times 10^6\left(\dfrac{d}{l}\right)^{3\,44}, \end{cases}$$

provided the ends of the tubes are sharp and the conditions within the range of values of R exhibited in fig 2.

The losses of energy are reduced by grinding the ends to a bell-mouth shape, although the effect becomes very small with the longest tubes.

The author is indebted to Professor A. H. Gibson, D.Sc., LL.D , for the facilities placed at his disposal in the Whitworth Engineering Laboratory of Manchester University, and to Mr R. Butt, M Sc., for assistance in both the experimental work and the calculation of results.

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(*Issued separately June* 30, 1936.)

V —The Revised Prepared System of the Quadratic Complex.

By **Professor H. W. Turnbull,** F.R S

(MS. received April 1, 1936 Read June 1, 1936)

INTRODUCTION

THE following is a sequel to two earlier papers by Dr J Williamson and the author, on "The Invariant Theory of the Quaternary Quadratic Complex," which appeared in the *Proceedings* in 1928 As a result of further study of two quadratics in n variables it appeared that the list of forms which constituted the starting-point of the former investigation was incomplete. The present work gives the prepared system on the basis of the revised data, and thus rectifies the earlier of the 1928 papers It will be convenient to refer to these three papers as I, II, and III, their full title and reference being given at the end of this investigation. I take this opportunity also of correcting the suffixes of an inequality in III, p. 21 (4), which should read

$$n > i_1 > i_2 > \ldots > i_\gamma > 0.$$

The notation of previous papers is, for the most part, adopted. This involves a possible confusion, in that the letters P and Q are used in I to denote variables (*cf.* § 1 (2) below), whereas in III they denote systems of forms. In what follows they denote variables, unless the context definitely states otherwise.

My best thanks are due to Dr J. Williamson for checking and improving a number of the following results.

§ 1. While the general argument of I, which refers the problem to that of two quadratics in six variables, is sound, the actual list on p. 76 of I is incomplete in the light of the results stated in III, pp. 16–21. The revised list, giving the irreducible system for two such quadratics

$$\left.\begin{aligned} \psi &= a_\xi^2 = b_\xi^2 = \ldots, \\ \Omega &= r_\xi^2 = s_\xi^2 = \ldots, \end{aligned}\right\} \qquad (1)$$

in six variables can at once be written down in the notation of III. It consists of P-forms and Q-forms. The P-forms are of type $(A_q R_i)_k^2$ with $k = q + i - 6 \geqslant 0$. For a reason to be given later, called the R_3-rule, the cases whenever $k > 2$ may be rejected. Here A_q denotes a convolution of q equivalent symbols, such as a, b, etc., while R_i denotes i such symbols r, s, etc. As before, there are four cogredient variables ξ, η, ζ, ω, which must ultimately be convolved in the manner

$$p=\xi\eta, \qquad P=\xi\eta\zeta, \qquad Q=\xi\eta\omega, \qquad \pi=\xi\eta\zeta\omega \quad . \quad . \quad (2)$$

But when, say, k of them are convolved indiscriminately the result is denoted by ϵ^k as in I, p. 76. The P-forms are

(i) $(A_q R_{6-q})^2$, seven invariants,
(ii) $\theta^2_{q\xi}$, six quadratics, namely, a^2_ξ, q^2_ξ, r^2_ξ,

where $q_\xi=(A_q R_{7-q}\xi)$, $q=2$, 3, 4, 5 as before, together with three new forms

$$\left.\begin{aligned}(H\xi\eta)^2&=(A_3R_5p)^2,\\ (K\xi\eta)^2&=(A_5R_3p)^2,\\ (L\xi\eta)^2&=(A_4R_4p)^2,\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

which incidentally define three new symbols, each of currency 2,

$$H=A_3R_5=\sum_R(A_3R_3)R_2=\sum_A(aR_5)A_2, \qquad K=A_5R_3, \qquad L=A_4R_4. \quad . \quad (4)$$

The Q-forms are of the type

$$\text{(iii)} \quad \phi=\underset{i_1}{q_1}\,\underset{i_2}{q_2}\;.\;.\;.\;\underset{i_\gamma}{q_\gamma},$$

where $\gamma=1$, 2, 3, 4, 5 and each q or i is a positive integer. More fully

$$\phi=(A_{q_1}\epsilon_1^{q_1})(A_{q_1}R_{i_1}\epsilon_2^{j_1})(A_{q_2}R_{i_1}\epsilon_3^{j_2})\;.\;.\;.\;(R_{i_\gamma}\epsilon_{2\gamma}^{i_\gamma}),$$

a product of 2γ factors, where

$$0 < q_1 < q_2 < \;.\;.\;. < q_\gamma < 6 > i_1 > i_2 > \;.\;.\;. > i_\gamma > 0, \quad . \quad (5)$$

and

$$q_1+i_1=6+j_1 \geqslant 6, \qquad q_2+i_1=6+j_2 \geqslant 6, \text{ etc.} \quad . \quad . \quad (6)$$

Thus the sets (4′), (5), (6) of I, p. 76, are retained, while (7) and (8) are discarded in favour of the present (3) and (4).

REDUCTIONS.

§ 2. In view of these more complicated forms it is necessary to re-examine the reduction processes of I, pp. 80–89. The two chief reductions may be called *the R_3-rule* and *the r-rule.*

The R_3-rule: *a form wherein three symbols r are convolved together with any three variables is reducible.*

For, as in I, p. 80 (5),

$$(D_3R_3)(R_3\epsilon\epsilon'\epsilon'')=6(D_3\epsilon\epsilon'\epsilon''), \quad . \quad . \quad . \quad . \quad (7)$$

where D_3 denotes any symbols or variables of currency 3, R_3 denotes three equivalent symbols r, r', r'', and $\epsilon\epsilon'\epsilon''$ any three of ξ, η, ζ, ω. Such an identity applies whenever $(R_3\epsilon\epsilon'\epsilon'')$ occurs (since the duplicate symbols of R_3 may also be convolved, by a fundamental property of quadrics), and thus it reduces the degree in r of any such form by three

This rule evidently reduces the above P-forms whenever $k > 2$, since

$$(A_qR_i)_k \equiv (A_qR_i\epsilon^k) = \sum_R (A_qR_{6-q})(R_k\epsilon^k). \quad . \quad . \quad . \qquad (8)$$

It equally reduces a Q-form whenever the sum of two adjacent suffixes exceeds 8, since this again implies a factor of the type just written down. Similarly $i_\gamma < 3$.

Furthermore, expressions such as

$$(ijkP), \qquad (aijk\pi), \qquad (2345\pi), \ . \quad . \qquad (9)$$

where i, j, $k=2$, 3, 4, 5, reduce by this rule. For, by I, p. 89, $(ijkP) = (\theta_i\theta_j\theta_kP)$, and

$$\theta_i = \sum_R (A_iR_{6-i})r = \sum_A (A_{i-1}R_{7-i})a. \quad . \quad . \qquad (10)$$

Hence, by keeping A_i, A_j, A_k explicitly convolved,

$$(ijkP) = \Sigma(A_i \ . \ . \ .)(A_j \ . \ . \ .)(A_k \ . \ . \ .)(rstP), \qquad (11)$$

where three symbols r, s, t are convolved with P, so that the R_3-rule applies. This reduces several forms retained in I and II.

The r-Rule.—The rule of I, p. 82, § 7 (i), by which a factor $(r\epsilon)(r\epsilon')$ is thrown out, continues to be valid. In forming the X system of paper III, however, the corresponding reduction which threw out $(a\epsilon)(a\epsilon')$ is valueless: and hence the third Lemma of III, p. 19, no longer applies in the more delicate search for the H system in which all the variables must be properly convolved.

As an immediate consequence of the r-rule we have

$$q_\gamma + i_\gamma = 6 \quad . \quad . \quad . \quad . \qquad (12)$$

for the final pair of suffixes in a Q-form; otherwise a factor $(r\epsilon)(r\epsilon')$ must appear, reducing the form.

THE Q-FORMS.

§ 3. The actual list of Q-forms, § 1 (4), can now be made. Besides the conditions (5) and (6) governing the indices, each pair of adjacent indices, by the r- and the R_3-rules, must add up to 6, 7, or 8 except the last, which adds to 6 only. The final suffix $i_\gamma = 1$ or 2. The notation, such as $\{ALR\}$ for ${}^{2\;4}_{\;\;4\;2}$ will be explained in the theorem following:—

$\gamma=1$ $\quad \begin{matrix}4&\\&2\end{matrix}=\{a_4R_2\}$ $\qquad \gamma=5$ $\quad \begin{matrix}1&&2&&3&&4&&5&\\&5&&4&&3&&2&&1\end{matrix}=\{a2345r\}.$

$\gamma=2$

$\begin{matrix}2&&4&\\&4&&2\end{matrix}$ $\{ALR\}$

$\begin{matrix}3&&4&\\&4&&2\end{matrix}$ $\{aRL\epsilon\}$

$\begin{matrix}3&&4&\\&3&&2\end{matrix}$ $\{a4R\}$

$\begin{matrix}3&&5&\\&3&&1\end{matrix}$ $\{aKr\}$

$\begin{matrix}4&&5&\\&2&&1\end{matrix}$ $\{a_45r\}$

$\begin{matrix}4&&5&\\&3&&1\end{matrix}$ $\{a_4Kr\epsilon\}$

$\gamma=3$

$\begin{matrix}1&&2&&4&\\&5&&4&&2\end{matrix}=\{a2LR\}$

$\begin{matrix}1&&3&&4&\\&5&&3&&2\end{matrix}=\{aH4R\}$

$\begin{matrix}1&&3&&4&\\&5&&4&&2\end{matrix}=\{aHLR\epsilon\}$

$\begin{matrix}1&&3&&5&\\&5&&3&&1\end{matrix}=\{aHKr\}$

$\begin{matrix}2&&3&&4&\\&4&&3&&2\end{matrix}=\{A34R\}$

$\begin{matrix}2&&3&&4&\\&5&&3&&2\end{matrix}=\{AH4R\epsilon\}$

$\begin{matrix}2&&3&&4&\\&5&&4&&2\end{matrix}=\{\epsilon AHLR\epsilon\}$

$\begin{matrix}2&&3&&5&\\&4&&3&&1\end{matrix}=\{A3Kr\}$

$\begin{matrix}2&&3&&5&\\&5&&3&&1\end{matrix}=\{AHKr\epsilon\}$

$\begin{matrix}2&&4&&5&\\&4&&2&&1\end{matrix}=\{AL5r\}$

$\begin{matrix}2&&4&&5&\\&4&&3&&1\end{matrix}=\{ALKr\epsilon\}$

$\begin{matrix}3&&4&&5&\\&3&&2&&1\end{matrix}=\{a45r\}$

$\begin{matrix}3&&4&&5&\\&4&&2&&1\end{matrix}=\{aL5r\epsilon\}$

$\begin{matrix}3&&4&&5&\\&4&&3&&1\end{matrix}=\{\epsilon aLKr\epsilon\}$

$\gamma=4$

$\begin{matrix}1&&2&&3&&4&\\&5&&4&&3&&2\end{matrix}=\{a234R\}$

$\begin{matrix}1&&2&&3&&5&\\&5&&4&&3&&1\end{matrix}=\{a23Kr\}$

$\begin{matrix}1&&2&&4&&5&\\&5&&4&&2&&1\end{matrix}=\{a2L5r\}$

$\begin{matrix}1&&2&&4&&5&\\&5&&4&&3&&1\end{matrix}=\{a2LKr\epsilon\}$

$\begin{matrix}1&&3&&4&&5&\\&5&&3&&2&&1\end{matrix}=\{aH45r\}$

$\begin{matrix}1&&3&&4&&5&\\&5&&4&&2&&1\end{matrix}=\{aHL5r\epsilon\}$

$\begin{matrix}1&&3&&4&&5&\\&5&&4&&3&&1\end{matrix}=\{\epsilon aHLKr\epsilon\}$

$\begin{matrix}2&&3&&4&&5&\\&4&&3&&2&&1\end{matrix}=\{A345r\}$

$\begin{matrix}2&&3&&4&&5&\\&5&&3&&2&&1\end{matrix}=\{AH45r\epsilon\}$

$\begin{matrix}2&&3&&4&&5&\\&5&&4&&2&&1\end{matrix}=\{\epsilon AHL5r\epsilon\}$

$\begin{matrix}2&&3&&4&&5&\\&5&&4&&3&&1\end{matrix}=\{AHLKr\epsilon\epsilon\epsilon\}$

THEOREM I.—*Each of the above thirty-three Q-forms is expressible as a generalized Jacobian in two or more of the P-forms.*

Proof—This is a matter of detailed verification. For example, consider the form $\begin{matrix}2&&4&\\&4&&2\end{matrix}$ which has five symbolic factors, say,

$$g_1g_2g_3g_4g_5=(A_2\epsilon^2)(A_2R_4)(R_4A_4\epsilon_1^2)(A_4R_2)(R_2\epsilon_2^2). \qquad (13)$$

Put $A_4R_4=L$ and move R_4 from the second bracket factor g_2 to the fourth g_4 by a fundamental identity. On rejecting terms of higher currency in a or r we get

$$\Omega_{CB}(A_2\epsilon^2)(A_2C_2R_2)(L\epsilon_1^2)(B_2R_4)(R_2\epsilon_2^2), \qquad B_2C_2=A_4,$$

where Ω_{CB} denotes the six-term determinantal permutation of C_2, B_2: or, let us say,

$$(A_2LR_2)(A_2\epsilon^2)(L\epsilon_1^2)(R_2\epsilon_2^2).$$

Let A without a suffix stand for A_2 (that is, two symbols a), and R likewise for R_2, and the whole result be called

$$\{ALR\}=(ALR)(A\epsilon^2)(L\epsilon_1^2)(R_2\epsilon_2^2). \tag{14}$$

Such a form, with all its different symbols A, L, R convolved in one factor, and thereafter occurring separately attached to variables ϵ, is called a *generalized Jacobian.* Since each Q-form is amenable to this treatment, this proves the theorem.

Inspection shows that each Q-form is characterized by a zigzag of indices $\begin{smallmatrix}1&&2&&4&\\&5&&4&&2\end{smallmatrix}$. When this is, so to say, stratified in the manner ////, these strata 1, $\begin{smallmatrix}&2\\5&\end{smallmatrix}$, $\begin{smallmatrix}&4\\4&\end{smallmatrix}$, 2 denote the symbols a, θ_2 or 2, L, R in exactly the same order as they occur in the Jacobian form $\{a2LR\}$. This affords a useful check on the construction of the list. Stratification such as \\\ always produces pairs of integers whose sum is 6 or 7, as inspection again shows. In each case where it is 7 an ϵ is inserted within the brackets { }. The position of the ϵ is a matter of convention; and the reason for this is best seen by an example:—

Consider $\begin{smallmatrix}3&&4&&5&\\&4&&3&&1\end{smallmatrix}$. The stratification indicates α or A_3, L, K, r for its symbols, and ϵ, ϵ for the variables, arising from 7 occurring twice. Write the form as

$$g_1g_2g_3g_4g_5g_6g_7=g_1g_2g_3(A_4R_3\epsilon_1)(R_3A_5\epsilon_2^2)(A_5r)g_7.$$

Move R_3 from g_4 to g_6, reject terms of higher currency in a or r, and write K for R_3A_5. Then we get

$$(A_3\epsilon^3)(A_3R_4\epsilon)(R_4A_4\epsilon^2)(A_4rK\epsilon_1)(K\epsilon_2^2)(r\epsilon_3),$$

where, as always, the variables ϵ, ϵ^2, . . . are arbitrary. Now move A_4 from g_4 to g_2 and we get

$$\Omega_{ll'}(A_3\epsilon^3)(l\epsilon)(L\epsilon^2)(A_3l'rK\epsilon_1)g_6g_7,$$

where $ll'=L=A_4R_4$. This may be written

$$(\epsilon LA_3rK\epsilon_1)(A_3\epsilon^3)(L\epsilon^2)(K\epsilon_1^2)(r\epsilon_2)\equiv\{\epsilon\alpha LKr\epsilon_1\}, \quad . \quad . \tag{15}$$

where $\alpha=A_3$. Similar treatment applies to each of the Q-forms.

PRINCIPAL AND SECONDARY FACTORS.

§ 4. Reference to the P-forms now shows that we are dealing with symbols each of currency one, each of currency two, and

$$\left.\begin{array}{l} a,\ 2,\ 3,\ 4,\ 5,\ r, \\ A,\ H,\ K,\ L,\ R, \\ \alpha=A_3, \qquad \alpha_4=A_4, \end{array}\right\} \quad . \quad . \qquad (16)$$

of currency three and four respectively. Each Q-form consists of a factor containing two or more distinct symbols, a, . . , H, . ., A_4, followed by factors with the same symbols occurring singly

We shall call the former the *principal factor* and all other factors *secondary*.

Inspection of the list shows that principal factors fall into four classes:

$$\text{eighteen forms } \phi_0=(ALR), \quad (\alpha 4R), \quad (a2LR), \quad \text{etc.}, \qquad . \quad (17)$$

each factor containing two, three, . , or six symbols with total currency 6;

$$\text{ten forms} \qquad \phi_1=(\alpha RL\epsilon), \quad (\alpha_4 rK\epsilon), \quad \text{etc}, \qquad . \qquad (18)$$

with symbols of total currency 7 and therefore *one* variable $\epsilon=\xi$, η, ζ, or ω besides;

$$\text{four forms} \qquad \phi_2=(\epsilon L\alpha rK\epsilon), \quad \text{etc.}, \qquad . \qquad (19)$$

with symbols of total currency 8 and therefore *two* variables ϵ besides; and

$$\text{one form} \qquad \phi_3=(AHKLr\epsilon\epsilon\epsilon) \quad . \quad . \quad . \quad . \quad . \quad (20)$$

with a total currency 9 and *three* variables ϵ.

Here ϕ_i stands for a form which may be expanded as follows:—

$$\left.\begin{array}{l} \phi_1=(\alpha RL\epsilon)=(\alpha Rl)(l'\epsilon)-(\alpha Rl')(l\epsilon), \\ \phi_2=\Sigma\pm(\epsilon l)(l'\alpha rk)(k'\epsilon), \end{array}\right\} \quad . \quad . \quad . \quad (21)$$

a series of four terms wherein ll' and kk' are separately permuted, and

$$\phi_3=\Sigma\pm(Ahklr)(h'\epsilon_1)(k'\epsilon_2)(l'\epsilon_3), \quad . \quad . \quad . \quad (22)$$

a series of eight terms wherein hh', kk', ll' are separately permuted.

§ 5. To form the desired complete system we must combine products of all factors, principal or secondary, in such a way as to convolve the variables ϵ into their appropriate groupings, § 1 (2). The investigation is simplified by the following theorem.

THEOREM 2 —*A form which includes two principal factors is reducible.*

Proof.—Such a form is an aggregate of terms such as

$$\psi=(\ .\ .\ R)(.\ .\ .\ S)(R\epsilon')(S\epsilon'')\ .\ .\ .,$$

where R is of currency 1 or 2 in the symbols r, and so is S, while ϵ' and ϵ'' denote variables of corresponding currency. Let Ω denote the determinantal permutation of the symbols S, R belonging to the second and third factors of ψ. Then

$$\Omega\psi=\psi+\Sigma\chi,$$

where every term χ must contain at least one symbol s repeated in the third and fourth factors, and hence each χ reduces by the r-rule. Also $\Omega\psi$ reduces because it involves the convolution RS, which is of higher currency in r than that of ψ Hence ψ is expressed in reducible terms.

§ 6. It follows that the complete system will consist of two classes of form, let us say *principal forms* Θ, involving one principal factor, and *secondary forms* Φ with no principal factor. Each such form Θ or Φ will be composed of symbols (16) in duplicate The above theorem shows that, of the symbols r, R, *one* only may occur; and if so, it occurs in the principal factor of Θ, with its duplicate in a later secondary factor Two symbols r (or two R) cannot occur in secondary factors (*cf* I, p. 82, § 7 (1)).

Next, the results of I, p. 83, §§ 8–10, demonstrating the interchangeability of symbols θ'_i, θ_i, are applicable also to the present case, which includes principal factors. Furthermore, it will now be shown that they apply also to the symbols H, K, L, namely, that any two single equivalent symbols H, H' are interchangeable. Let $\theta_{ij}=A_iR_j$ (so that when $i+j=7$ this includes the case θ_i), and suppose that $\theta'_{ij}=B_iS_j$ is an equivalent symbol.

THEOREM —

$$(\theta_{ij}\lambda)(\theta'_{ij}\mu)-(\theta'_{ij}\lambda)(\theta_{ij}\mu)\equiv 0. \quad . \qquad . \qquad . \quad (23)$$

Proof —Write the first term of (23) as

$$\binom{k+l}{k}^2(A_kR_j)(A_l\ .\ .\ .)(B_kS_j)(B_l\ .\ .\ \ )[A_{k+l},\quad B_{k+l}],$$

where $A_k\,A_l=A_{k+l}=A_i$, and similarly for B. Then the second term of (23) is evidently obtained by interchanging A_l and B_l of the second and fourth symbolic factors. The left member of (23) is therefore equivalent to

$$\sum_{m>0}(A_kR_j)(A_mB_l\ .\ .\ .)(B_kS_j)(A_n\ .\ .\ .)[\ .\quad .,\quad .\ .\ .],$$

by a Sylvester identity (IV, p. 93), where $A_mA_n=A_l$. Each such term is equivalent to one with a second R_j replacing the S_j in the third factor, together with reducible terms involving r to higher currency than j Now consider the expression

$$X=(A_kR_j)[\Omega(A_mB_l\ .\ .\ .)(B_kR_j)](A_n\ .\ .\ .)[A_{k+l},\quad B_{k+l}], \qquad . \quad (24)$$

where Ω denotes the determinantal permutation of the $(m+l+k)$ equivalent symbols of A and B. If A_p is part of A_m, we have

$$(A_kR_j)(A_pB_{k-p}R_j)[A_{k+l},\ .\ .\ .]=0, \qquad p>0,$$

as is seen by interchanging the $(k+p)$ symbols a in all ways and adding the results. Hence in X the matrix A_m is static, and Ω affects B_l, B_k

only. Since the duplicates of these $(l+k)$ symbols occur convolved in B_{k+l}, each non-zero term of X is equivalent to the whole expression—that is, the whole is equivalent to the left member of (23). But, again, the operation Ω immediately raises the currency to $(m+l+k)$ in symbols of a, and therefore reduces the form. This proves the statement (23).

Corollary.—If H' is partly or wholly convolved with H the form reduces. Similarly for K and L.

For such a form is equivalent to a term of the series (24), which term has just been demonstrated to be reducible.

THE PREPARED SYSTEM OF SECONDARY FACTORS.

§ 7 To obtain the Prepared System it is necessary to introduce the variables ξ, $\xi\eta$, $\xi\eta\zeta$, $\xi\eta\zeta\omega$ properly convolved. These enter into principal as well as secondary factors, excluding of course the ϕ_0 which contain symbols only. On the basis of the results in I, p. 90, the following are the possible secondary factors:—

$$\begin{array}{l|l}
F_0 & (A_i R_{6-i})^2, \quad i=0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6, \qquad r_\zeta r_\omega = |\delta|; \\
F_1 & a_x,\ 2_x,\ 3_x,\ 4_x,\ 5_x,\ r_x,\ (Ap),\ (Hp),\ (Kp),\ (Lp),\ (Rp),\ (\alpha P),\ (\alpha_4\pi); \\
F_2 & (a2),\ (a3),\ (a4),\ (a5),\ (r2),\ (r3),\ (r4),\ (r5),\ (ij),\ (ar), \\
 & (Ar),\ (A3),\ (A4),\ (A5),\ (aH),\ (aK),\ (aL),\ (aR), \\
 & (\alpha 4),\ (\alpha 5),\ (\alpha r),\ (AK),\ (AL),\ (AR), \\
 & (AH)',\ (AK)',\ (AL)',\ (AR)',\ (\alpha K)',\ (\alpha L)',\ (\alpha R)'; \\
F_3 & (aij),\ (air),\ (A34),\ (A35),\ (A45),\ (A3r),\ (A4r),\ (A5r)
\end{array}$$

Here $i, j = 2, 3, 4, 5$, so that (ij) stands for six distinct possibilities, with $i \neq j$. In each of the fifteen factors of the first row of F_2 a variable $p = \xi\eta$ is understood. Thus $(ij) = (\theta_i \theta_j p)$ The eight factors of the second row involve $P = \xi\eta\zeta$. The next six involve π. The next four, marked with an accent, involve both ξ and P (*cf.* I, p. 89): thus

$$(AH)' = (\xi AHP) = (\xi a)(a'HP) - (\xi a')(aHP),$$

while

$$(\alpha K)' = (\xi \alpha K\pi) = \Sigma(\xi a)(a'a''K\pi), \qquad (aa'a'' = \alpha). \quad . \quad . \quad (25)$$

Thus the three $(\alpha K)'$, $(\alpha L)'$, $(\alpha R)'$ involve ξ and π. Finally, the ten F_3 factors (aij), (air) each involve P, and the remaining six involve π; so that

$$(aij) = (aijP), \qquad (A5r) = (A5r\pi). \quad . \quad . \quad (26)$$

The above list is made synthetically by combining the symbols a, i, r, A, H, K, L, R, α, α_4 with the four variables ξ, p, P, π in every possible way, subject to the reducibility conditions already proved. Thus the symbol H is never convolved with r or R (by the R_3-rule) and never with h, k, or l; similarly for K, L. The R_3-rule excludes also Ri, ijk, while the convolutions $A2$, $\alpha 2$, $\alpha 3$ are excluded, since each raises the currency in a: thus $A2$ has three symbols a convolved (by (10)), and therefore raises the currency from two (that of A and also of θ_2) to three, without lowering the currency (five) in r Similarly the currency of $\alpha 2$, and of $\alpha 3$, rises from three to four. Finally (HA) is omitted from the list, since its currency in a is four, while that of H is three and of A is two.

FURTHER IDENTITIES.

§ 8. Before considering the list of *principal* factors it is convenient to state certain relations which indicate reducible terms. The following combinations reduce to terms either containing an invariant factor or else having higher currency in a, r, or both:—

$$\left.\begin{array}{l} H\alpha,\ HA,\ H2,\ H3,\ KR,\ K4,\ K5,\ L3,\ L4,\ A2,\ \alpha 2,\ \alpha 3,\ R5; \\ (A3, A3),\quad (\alpha 4, \alpha 4),\quad (LA, LA),\quad (LR, LR),\quad (L\alpha, L\alpha), \\ (a2, a2),\quad (23, 23),\quad (34, 34),\quad (45, 45),\quad (5r, 5r). \end{array}\right\} \quad . \qquad (27)$$

This notation means that a form containing, *e g.*, $L4$ convolved once and not necessarily twice reduces; but if LA occurs it must occur convolved twice in order to reduce. The proofs follow the lines of I, p. 84, and vary in complexity. The proofs for (34, 34) are given in illustration below. There are also several useful relations :—

$$\left.\begin{array}{llll} & (a2, a2)\equiv\tfrac{1}{3}(H, H), & (r5, r5)\equiv\tfrac{1}{3}(K, K), & \\ (L2, 23)\equiv 0, & (L5, 45)\equiv 0, & (A3, 23)\equiv 0, & (R4, 45)\equiv 0, \\ (a2, aij)\equiv 0, & (A3, Aij)\equiv 0, & (aH, a2)\equiv 0, & (rK, r5)\equiv 0, \end{array}\right\} . \qquad . \quad (28)$$

the first of these meaning that a form with $a2$ convolved twice is equivalent to the form with H substituted for $a2$: or, again, a form with $L2$ and 23 occurring separately is reducible. Furthermore,

(i) *The convolution $a2$ and the symbol A are interchangeable;*

(ii) *A form containing both R and 5 is reducible;*

(iii) *A form containing K twice in the secondary factors is reducible.*

Proofs.—(i) follows since Aa, $A2$ are both reducible.

(ii) The form $\{R\ .\ .\ .\}(5\epsilon)$ reduces by moving 5 to the principal factor, so obtaining $R5(\equiv 0)$ or displacing a symbol r to the secondary factor (*cf* § 5).

(iii) Replace K, K by $5r$, $5r$, and apply the r-rule.

Proof that $(34, 34)\equiv 0$. Such a form is equivalent to

$$(a'a''R_4)(ab)(b'BR_3)(aa''R_4)(a'b')(bBR_3),$$

where $3\equiv A_3R_4$, $4\equiv B_4R_3$, $aa'a''=A_3$, $bb'B=B_4$. Convolve a', b' from g_1, g_3 into g_1: and then, in such terms as have a symbol of R_4 moved to g_3, convolve R_4 in g_3, rejecting terms of higher currency R_5 Repeat the process by convolving a, b from g_4, g_6 into g_4; and reject obviously reducible terms in the expanded product. There remain only terms of the type

$$(A'R_4)(A)(a''BR_3)(Aa''R_3)(A')(BR_4),$$

where $A'=a'b'$, $A=ab$. This type reduces at once by convolving $a''B$ from g_4, g_6 into g_4.

PREPARED SYSTEM OF PRINCIPAL FACTORS.

§ 9. It now remains to insert the variables in the principal factors ϕ_i. Evidently the eighteen forms ϕ_0 need no further treatment, while the forms ϕ_1, ϕ_2, ϕ_3 behave as if they had currency one, two, three respectively. The complete list is as follows.—

Principal Factors

ϕ_0 | (α_4R), (ALR), $(\alpha 4R)$, (αKr), (α_45r), $(a2LR)$,
$(aH4R)$, $(aHKr)$, $(A34R)$, $(A3Kr)$, $(AL5r)$, $(\alpha 45r)$,
$(a234R)$, $(a23Kr)$, $(a2L5r)$, $(aH45r)$, $(A345r)$, $(a2345r)$.

ϕ_1 (i) | (αRLX), $X=\xi$, ap, $2p$,
(ii) | (α_4KrX), $X=ap$, $2p$, $3p$, $a2P$, $a3P$, AP, $Ar\pi$,
(iii) | $(aHRL\xi)$, $(aHRLap)$,
(iv) | $(A4RH\xi)$, $(A4RH4p)$,
(v) | $(ArKHX)$, $X=\xi$, $4p$, $5p$, rp,
(vi) | $(ALrKX)$, $X=\xi$, ap, $2p$, rp, AP, $a2P$, arP, $Ar\pi$,
(vii) | $(\alpha 5rLX)$, $X=\xi$, ap, $2p$, $5p$, rp, arP, $a5P$,
(viii) | $(a2LKrX)$, $X=\xi$, ap, $2p$, rp, $a2P$, arP,
(ix) | $(Ar45HX)$, $X=\xi$, $4p$, $5p$, rp,
(x) | $(aH5rLX)$, $X=\xi$, ap, $5p$, rp, AP, $a5P$, arP,

ϕ_2 (i) | $(\xi AHRL\xi)(Ap)(Hp)(Rp)(Lp)$,
(ii) | $(\xi L\alpha rKX)$, $X=\xi$, ap, $2p$, rp, $a2P$, arP,
$(paL\alpha rKX)$, $X=ap$, $2p$, rp, $a2P$, arP,
$(p2L\alpha rKX)$, $X=2p$, rp, $a2P$,
$(prL\alpha rKa2p)$,
(iii) | $(XHKarLY)$, $(X, Y)=(\xi, \xi)$, (ξ, ap), (ξ, rp),
(pa, ap), (pa, rp), (ξ, arP), (pa, arP),
(iv) | $(XHA5rLY)$, $(X, Y)=(\xi, \xi)$, $(\xi, 5p)$, (ξ, rp),
$(5p, 5p)$, $(5p, rp)$,

ϕ_3 | $(AHLKr\xi\xi\xi)(Ap)(Hp)(Kp)(Lp)r_\xi$,
" " " " (LrP),
$(AHLKr\xi\xi\ rp)(Ap)(Hp)(Kp)(Lp)$.

In the above list the results under ϕ_1 are obtained by treating each of the ten cases separately, since two such factors cannot occur simultaneously

(§ 5). The form ϕ_1 is regarded as a symbol of currency one, which may be combined exactly as the a, i, r of (16).

For example, under ϕ_1 (iv), write

$$\phi = (A4Rh)h' - (A4Rh')h,$$

then ϕ_ξ, (ϕip), (ϕAP), (ϕiaP), etc., are possibilities. But all except ϕ_ξ, $(\phi 4p)$ reduce (these being the two retained in the list), as indeed appears on detailed examination in the light of § 7. Thus $(\phi 3p)$ gives

$$(A4Rh)(h'3p) - (A4Rh')(h3p) = (A4R3)(h'hp) + (A4R.H3.p),$$

where the first term on the right is expressed by a ϕ_0 factor with (Hp), and the second reduces with $H3$ (§ 8). Each ϕ_i is considered reduced if expressed in terms of lower suffix i.

Similar treatment avails for the ϕ_2 and ϕ_3, on the assumption that their variables ϵ link in the above manner. Thus ϕ_2 (iv) gives, say,

$$\phi\phi' = \Sigma h(h'A5l)l'$$

and leads to $\phi_\xi\phi'_\xi$, $(\phi ip)(\phi' jp)$, etc., five only of which are possibly irreducible And so on. This assumption rests on the easily proved fact that *no two variables* ϵ *contained in a form* ϕ_2 *or* ϕ_3 *may be convolved together*.

For instance, ϕ_3 as defined in (22) is readily seen to alternate in the four symbols A, H, K, L (rejecting reducible terms). Now if $\phi_3 = \phi(\epsilon_1, \epsilon_2, \epsilon_3)$, it follows at once from (22) that $\phi(\epsilon_1, \epsilon_2, \epsilon_3) - \phi(\epsilon_2, \epsilon_1, \epsilon_3) \equiv 0$, and hence ϕ_3 reduces if the variables $\epsilon_1\epsilon_2$ are convolved together. Similarly for all other cases.

THE COMPLETE SYSTEM.

§ 10. The complete system is now constructed as in II from all suitable combinations of secondary factors, subject to the rules of § 8, with or without the principal factors. Each principal factor is separately combined with all suitable sets of secondary factors. It appears that only the three forms indicated survive under ϕ_3, all others being definitely reducible, while ϕ_2 (i) yields one possibility, as shown in the above prepared list Under ϕ_0 the last case $(a2345r)$ also yields but one result, the Jacobian

$$S = (a2345r)a_\xi 2_\xi 3_\xi 4_\xi r_\xi.$$

As for the other cases of principal factor, with very few exceptions they admit secondary factors of types F_1 and F_2 only; and this fact materially simplifies the investigation.

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(*Issued separately June* 30, 1936)

VI.—On Singular Pencils of Zehfuss, Compound, and Schläflian Matrices. By W. Ledermann.* *Communicated by* Professor H. W. TURNBULL, F.R.S (With Five Diagrams.)

(MS received February 14, 1936 Read June 1, 1936)

§ 1. INTRODUCTION

IN this paper the canonical form of matrix pencils will be discussed which are based on a pair of direct product matrices (Zehfuss matrices), compound matrices, or Schläflian matrices derived from given pencils whose canonical forms are known

When all pencils concerned are non-singular (*i.e* when their determinants do not vanish identically), the problem is equivalent to finding the elementary divisors of the pencil. This has been solved by Aitken (1935), Littlewood (1935), and Roth (1934). In the singular case, however, the so-called *minimal indices* or *Kronecker Invariants* have to be determined in addition to the elementary divisors (Turnbull and Aitken, 1932, chap. IX). The solution of this problem is the subject of the following investigation.

The method employed is that of the *principle of vector chains* which was first used in this connection by Turnbull (1935). Our process will also show many analogies to Aitken's "principle of unit step" (Aitken, 1935, p. 367)

Let $\rho A+\sigma B$ be a pencil of type $m \times n$, *i e.* with m rows and n columns. It is then possible to find two non-singular constant matrices P of degree m and R of degree n such that

$$P(\rho A+\sigma B)R=\begin{bmatrix} 0_{e,f} & 0 \\ 0 & \rho A+\sigma B \end{bmatrix}, \quad . \quad . \qquad (1a)$$

there being $e \geqq 0$ zero rows and $f \geqq 0$ zero columns, and where

$$(\rho A_1+\sigma B_1)=\operatorname{diag}(L_{n_1}, L_{n_2}, \ldots, L_{n_a}, M_{m_1}, M_{m_2}, \quad , M_{m_b}, N_{p_1}, N_{p_2}, \ldots, N_{p_c}, Z), \quad (1b)$$

where the symbols on the right-hand side have the following meaning (Ledermann, 1935, p 94):—

$$L_i=\rho F_i+\sigma G_i=\begin{bmatrix} \rho & \sigma & . & . \\ . & \rho & \sigma & . \\ . & & . & . \\ . & . & \rho & \sigma \end{bmatrix}_{i,\,i+1} \quad . \quad . \quad . \qquad (2a)$$

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[*] The material of this paper was used as part of a thesis which was submitted for the Ph D degree at St Andrews in March 1936. I wish to express my thanks to Dr A C. Aitken for many suggestions and for his help in preparing this paper.

is a pencil with i rows and $i+1$ columns, *e g.*

$$L_4=\begin{bmatrix} \rho & \sigma & \cdot & \cdot & \cdot \\ & \rho & \sigma & & \\ \cdot & & \rho & \sigma & \cdot \\ \cdot & & & \rho & \sigma \end{bmatrix}$$

and

$$M_j=\rho G_j'+\sigma F_j'=\begin{bmatrix} \sigma & \cdot & \\ \rho & \sigma & \cdot \\ & \rho & \cdot \\ \cdot & \cdot & \sigma \\ & \cdot & \rho \end{bmatrix}_{j+1,\,j} \tag{2b}$$

has $j+1$ rows and j columns. The pencil N_k is defined as

$$N_k=\rho I_k+\sigma U_k=\begin{bmatrix} \rho & \sigma & & & \\ & \rho & \sigma & & \\ & & & & \\ & & & \rho & \sigma \\ & & & & \rho \end{bmatrix}_{k,\,k} \quad . \quad . \quad (3)$$

where I_k is the unit matrix and U_k the auxiliary unit matrix of degree k (Turnbull and Aitken, 1932, p. 62). Each submatrix N_k represents an infinite latent root, while L_i and M_j correspond to the linear relations between the rows and between the columns of the pencil. Finally,

$$Z=\rho C+\sigma D$$

is a non-singular pencil, in which $|D|\neq 0$. It will be observed that this canonical form differs somewhat from that usually given in the literature. But it is easy to see that the modifications are irrelevant; in particular we note that the submatrices L_j and M_j are connected by

$$M_j=J_{j+1}L_j'J_j,$$

where

$$J_j=\begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ \cdots & \cdots & \\ 1 & \cdot & \cdot \end{bmatrix}_{jj} \quad . \quad . \quad . \quad . \quad (4)$$

In order to state the result (1) more conveniently, we introduce the "*direct sum*" of matrices, viz.

$$A\dotplus B=\begin{bmatrix} A & \cdot \\ \cdot & B \end{bmatrix},$$

and in general

$$A_1\dotplus A_2\dotplus \ . \ . \ . \ \dotplus A_h=\sum_{i=1}^{h}{}^{\cdot} \, . \, A_i=\operatorname{diag}(A_1, A_2, \ldots, A_h).$$

We shall also use the abbreviation

$$A\dotplus A=2\,.\,A, \qquad A\dotplus A\dotplus A=3\,.\,A, \ \ldots, \text{ etc.,}$$

which, of course, has to be carefully distinguished from scalar multiples of A which are written

$$2A, \qquad 3A, \ldots$$

Further, it is useful to introduce two symbols to the following effect: if A is a matrix with r rows and s columns, we put

$$L_0 + A = [\quad A]; \qquad 2\,.\,L_0 + A = [.\quad .\quad A], \text{ etc.,} \qquad . \qquad (5a)$$

$$M_0 + A = \begin{bmatrix} . \\ A \end{bmatrix}; \qquad 2\,.\,M_0 + A = \begin{bmatrix} . \\ . \\ A \end{bmatrix}, \text{ etc.,} \qquad . \qquad (5b)$$

i.e. the term $L_0[M_0]$ means that A has to be augmented by a zero column [row]. The symbols L_0 and M_0 are, of course, not proper matrices.

In this notation we can write (1*a*) and (1*b*) as

$$P(\rho A + \sigma B)R = e\,.\,L_0 + f\,.\,M_0 \dotplus (\rho A_1 + \sigma B_1), \qquad . \qquad (6a)$$

$$\rho A_1 + \sigma B_1 = \sum_i . L_{n_i} \dotplus \sum_j . M_{m_j} \dotplus \sum_k . N_{p_k} \dotplus Z, \qquad . \qquad (6b)$$

or more briefly

$$\rho A + \sigma B \sim \sum_i . L_{n_i} \dotplus \sum_j . M_{m_j} \dotplus \sum_k . N_{p_k} + Z \qquad (7)$$

if we include the zero indices, if any, among the n_i and m_j and where $\sim$ indicates equivalence.

According to the general theory, Z may be transformed into the classical Weierstrassian form, viz.

$$Z = \sum_r . W_{h_r}(\alpha_r) \dotplus \sum_l . Q_{g_l},$$

where

$$W_h(\alpha) = (\alpha\rho + \sigma)I_h + \rho U_h' = \begin{bmatrix} \alpha\rho+\sigma & . & & . \\ \rho & \alpha\rho+\sigma & & \\ . & \cdots & & .. \\ . & . & \rho & \alpha\rho+\sigma \end{bmatrix}_{h,h} \qquad . \qquad (8)$$

and

$$Q_g = \sigma I_g + \rho U_g' = \begin{bmatrix} \sigma & . & . \\ \rho & \sigma & . \\ \cdots & \cdots & \\ . & \rho & \sigma \end{bmatrix}_{g,g} = W_g(0). \qquad . \qquad (9)$$

The quantities $\alpha_1, \alpha_2, \ldots$ are non-zero, but not necessarily distinct. Substituting in (7) we obtain

$$\rho A + \sigma B \sim \sum_i . L_{n_i} \dotplus \sum_j . M_{m_j} \dotplus \sum_k . N_{p_k} \dotplus \sum_h . Q_{g_h} \dotplus \sum_r . W_{h_r}(\alpha_r), \qquad (10)$$

where n_i and m_j are the two sets of minimal indices of the pencil $\rho A + \sigma B$ referring to the columns and rows respectively. They are the minimal degrees in ρ and σ of the column and row vectors which annihilate the pencil, and they may be positive or zero It is no loss of generality, however, to assume that no zeros occur among them, *i e.* that $e=f=0$ in (1*a*) (Ledermann, 1935, p. 93). But later on, when dealing with composite pencils, we shall see that zero values for n_i and m_j cannot be avoided.

It should be noted that direct summation of matrices is commutative if we do not distinguish between equivalent pencils (Littlewood, 1935, p 373), thus

$$\rho(C_1 \dotplus C_2)+\sigma(D_1 \dotplus D_2) \sim \rho(C_2 \dotplus C_1)+\sigma(D_2 \dotplus D_1),$$

i.e these two pencils have the same canonical form.

§ 2. VECTOR CHAINS AND CANONICAL FORM OF A PENCIL

In two papers by Turnbull (1935) and the author (1935) it has been shown how a pencil can be brought into its canonical shape by successively forming certain vector chains. We shall here use a generalisation of this principle, which has also been hinted at by Professor Turnbull Let

$$\rho A+\sigma B$$

be a pencil with m rows and n columns, and consider the equation

$$(\rho A+\sigma B)R=R^*L_{i-1}, \quad . \quad . \quad . \quad . \quad (\text{i}, a)$$

where L_{i-1} is defined in § 1 (2*a*). The matrix R is of type $n \times i$ and R^* is of type $m \times (i-1)$. Throughout in this section the columns of R and R^* will be linearly independent. Put

$$R=[r_1, r_2, \ldots, r_i] \quad \text{and} \quad R^*=[r_1^*, r_2^*, \ldots, r_{i-1}^*], \quad . \quad (1)$$

where the r's and the r^*'s are column vectors of n and m elements respectively Writing (i, *a*) in full, we have

$$(\rho A+\sigma B)[r_1, r_2, \ldots, r_i]=[r_1^*, r_2^*, \ldots, r_{i-1}^*]\begin{bmatrix} \rho & \sigma & . & . & . \\ . & \rho & \sigma & & \\ . & . & . & . & \cdots \\ & . & & \rho & \sigma \end{bmatrix}_{i-1, i}. \quad (2)$$

Hence

$$[\rho Ar_1+\sigma Br_1, \rho Ar_2+\sigma Br_2, \ldots, \rho Ar_i+\sigma Br_i]=[\rho r_1^*, \sigma r_1^*+\rho r_2^*, \ldots, r_{i-1}^*\sigma];$$

that is

$$Ar_1=r_1^*; \quad Ar_2=r_2^*; \quad \ldots; \quad Ar_{i-1}=r_{i-1}^*; \quad Ar_i=0. \quad . \quad (3)$$

$$Br_1=0; \quad Br_2=r_1^*; \quad \ldots; \quad Br_{i-1}=r_{i-2}^*; \quad Br_i=r_{i-1}^*. \quad (4)$$

From (3) and (4) we derive a *vector chain* † *of type L*:

$$0=Br_1; \quad Ar_1=Br_2; \quad Ar_2=Br_3; \quad \ldots; \quad Ar_{i-1}=Br_i; \quad Ar_i=0. \quad (\text{i}, b)$$

---

† In the two papers cited at the beginning of this paragraph row vectors are used instead of column vectors, which we here prefer merely for technical reasons. The fact that either row or column vectors have to be distinguished is certainly a disadvantage which is, however, shared by most of the other theories in this field.

On the other hand, suppose we have found i vectors $r_1, r_2, \ldots, r_i$ satisfying (i, *b*); we can then define the vectors $r_1^*, r_2^*, \ldots, r_{i-1}^*$ by (3) or (4) and get back to (i, *a*). The equations (i, *a*) and (i, *b*) are therefore equivalent. In the same manner we can deal with the other submatrices M_j, N_k, and Q_g that occur in the general canonical form § 1 (10)

Let

$$(\rho A+\sigma B)R=R^*M_j, \qquad \text{(ii, } a\text{)}$$

where R must be of type $n\times j$ and R^* of type $m\times(j-1)$ Put

$$R=[r_1, r_2, \ldots, r_j] \quad \text{and} \quad R^*=[r_0^*, r_1^*, \ldots, r_j^*]. \qquad (5)$$

Substituting (5) and § 1 (2*b*) in (ii, *a*), we get

$$(\rho A+\sigma B)[r_1, r_2, \ldots, r_j]=[r_0^*, r_1^*, \ldots, r_j^*]\begin{bmatrix} \sigma & \cdot & & \\ \rho & \sigma & \cdot & \\ \cdot & \rho & & \\ & & & \sigma \\ & & & \rho \end{bmatrix}_{j+1, j}.$$

Hence

$$Ar_1=r_1^*; \quad Ar_2=r_2^*, \quad \ldots, \quad Ar_{j-1}=r_{j-1}^*, \quad Ar_j=r_j^*; \qquad (6)$$

$$Br_1=r_0^*; \quad Br_2=r_1^*; \quad \ldots; \quad Br_{j-1}=r_{j-2}^*; \quad Br_j=r_{j-1}^*, \qquad (7)$$

whence we derive a *vector chain of type M*:

$$0\neq Br_1; \quad Ar_1=Br_2; \quad \ldots, \quad Ar_{j-1}=Br_j; \quad Ar_j\neq 0. \qquad \text{(ii, } b\text{)}$$

Next, consider

$$(\rho A+\sigma B)R=R^*N_k, \qquad \text{(iii, } a\text{)}$$

where R is of type $n\times k$ and R^* of type $m\times k$ Put

$$R=[r_1, r_2, \ldots, r_k] \quad \text{and} \quad R^*=[r_1^*, r_2^*, \ldots, r_k^*] \qquad (8)$$

and substitute (8) and § 1 (3) in (iii, *a*):

$$(\rho A+\sigma B)[r_1, r_2, \ldots, r_k]=[r_1^*, r_2^*, \ldots, r_k^*]\begin{bmatrix} \rho & \sigma & \cdot & \cdot & \\ \cdot & \rho & \sigma & \cdot & \cdot \\ \cdots & \cdots & \cdots & & \\ \cdot & & \cdot & \rho & \sigma \\ \cdot & & \cdot & \cdot & \rho \end{bmatrix}_{k, k}.$$

Hence

$$Ar_1=r_1^*; \quad Ar_2=r_2^*; \quad \ldots; \quad Ar_{k-1}=r_{k-1}^*; \quad Ar_k=r_k^*, \qquad (9)$$

$$Br_1=0; \quad Br_2=r_1^*; \quad \ldots; \quad Br_{k-1}=r_{k-2}^*; \quad Br_k=r_{k-1}^*. \qquad (10)$$

Equations (9) and (10) yield a *chain of type N*:

$$0=Br_1; \quad Ar_1=Br_2, \quad \ldots, \quad Ar_{k-1}=Br_k; \quad Ar_k\neq 0. \qquad \text{(iii, } b\text{)}$$

Finally, let

$$(\rho A+\sigma B)R=R^*Q_g, \qquad \text{(iv, } a\text{)}$$

where R is of type $n\times g$ and R^* of type $m\times g$. Put

$$R=[r_1, r_2, \ldots, r_g] \quad \text{and} \quad R^*=[r_1^*, r_2^*, \ldots, r_g^*]. \qquad (11)$$

On substituting (11) and § 1 (9) in (iv, *a*) we obtain

$$(\rho A+\sigma B)[r_1, r_2, \ldots, r_g]=[r_1^*, r_2^*, \ldots, r_g^*]\begin{bmatrix}\sigma & & & & \\ \rho & \sigma & & & \\ & \rho & \sigma & & \\ & & & \ddots & \\ & & & \rho & \sigma\end{bmatrix}_{g, g}.$$

Hence comparing coefficients of ρ and σ we get

$$Ar_1=r_2^*; \quad Ar_2=r_3^*; \quad \ldots; \quad Ar_{g-1}=r_g^*; \quad Ar_g=0, \qquad (12)$$

$$Br_1=r_1^*; \quad Br_2=r_2^*; \quad \ldots, \quad Br_{g-1}=r_{g-1}^*; \quad Br_g=r_g^* \qquad (13)$$

This gives rise to a *vector chain of type Q*:

$$0\neq Br_1; \quad Ar_1=Br_2; \quad Ar_2=Br_3, \quad \ldots; \quad Ar_{g-1}=Br_g; \quad Ar_g=0. \qquad (\text{iv}, b)$$

It will simplify the work if we write the chains (i, *b*), (ii, *b*), (iii, *b*), and (iv, *b*) in the following *standard form*, which enables us to deal with them more uniformly. Let

$$r_0^*=Br_1; \quad r_1^*=Ar_1=Br_2; \quad \ldots; \quad r_{p-1}^*=Ar_{p-1}=Br_p; \quad r_p^*=Ar_p \qquad (14)$$

According as r_0^* and r_p^* are zero or not, the vector chain (14) is of one of these four types, viz of

$$\left.\begin{array}{llll}\text{type } L, \text{ if} & r_0^*=0; & r_p^*=0 & (\text{i}, b),\\ \text{type } M, \text{ if} & r_0^*\neq 0; & r_p^*\neq 0 & (\text{ii}, b),\\ \text{type } N, \text{ if} & r_0^*=0; & r_p^*\neq 0 & (\text{iii}, b),\\ \text{type } Q, \text{ if} & r_0^*\neq 0; & r_p^*=0 & (\text{iv}, b).\end{array}\right\} \qquad (15)$$

They correspond to the submatrices L_{p-1}, M_p, N_p, Q_p respectively The number p is called the *length* of the chain (4). The matrix R always has the columns $r_1, r_2, \ldots, r_p$, while the columns of R^* are the non-zero vectors out of the set $r_0^*, r_1^*, \ldots, r_{p-1}^*, r_p^*$.

We add a simple example in order to show the principal idea of our method. Supposing we have determined three pairs of matrices R_1, R_1^*, R_2, R_2^*; R_3, R_3^* which satisfy (i, *a*), (ii, *a*), and (iv, *a*) respectively. We can then comprehend these equations in

$$(\rho A+\sigma B)[R_1, R_2, R_3]=[R_1^*, R_2^*, R_3^*](L_\epsilon \dotplus M_\eta \dotplus Q_g).$$

Now, if the numbers of columns in R_1, R_2, R_3 and R_1^*, R_2^*, R_3^* are such that the matrices $[R_1, R_2, R_3]$ and $[R_1^*, R_2^*, R_3^*]$ are square and *non-singular*, then we have proved that the canonical form of $\rho A+\sigma B$ is $L_\epsilon \dotplus M_\eta \dotplus Q_g$. We shall therefore aim at constructing as many matrices R_1, R_2, . . ; R_1^*, R_2^*, . . . as are necessary to build up two non-singular matrices $[R_1, R_2, \ldots]$ and $[R_1^*, R_2^*, \ldots]$.

§ 3. PRELIMINARY REMARKS ABOUT DIRECT PRODUCTS.

If $A=[a_{ij}]$ is a matrix of type $m\times n$ and $B=[b_{kl}]$ a matrix of type $p\times q$, then their direct product

$$A\times B=\begin{bmatrix} a_{11}B & . & . & a_{1n}B \\ a_{m1}B & . & . & a_{mn}B \end{bmatrix} \quad . \quad . \quad (1)$$

is a matrix of type $mp\times nq$. It can be defined for any two matrices. Direct multiplication obeys the associative and the distributive law, viz.

$$\begin{aligned}&(A\times B)\times C=A\times(B\times C)=A\times B\times C,\\ &A\times(B+C)=(A\times B)+(A\times C),\\ &(A+B)\times C=(A\times C)+(B\times C),\end{aligned}$$

as can easily be verified.

There is also a distributive law connecting direct addition and direct multiplication, viz.

$$(A_1\dot{+}A_2)\times B=(A_1\times B)\dot{+}(A_2\times B). \qquad (2)$$

Again,

$$(A\times B)'=(A'\times B'). \quad . \quad . \quad . \qquad (3)$$

Both these rules readily follow from the definition (1). It should be noted that no simple relation like (2) exists when the second factor is a direct sum.

The most important property of direct multiplication is the multiplicative law:

$$(R\times S)(A\times B)=(RA\times SB). \quad . \quad . \quad . \qquad (4)$$

If A and B are square matrices of non-singular determinants, so is $A\times B$. This follows from (4) by putting $R=A^{-1}$ and $S=B^{-1}$ The right-hand side then becomes a unit matrix and neither of the factors on the left-hand side can be of zero determinant.

Consider the transformation of two sets of variables:

$$\left.\begin{aligned}\rho_i&=\sum_{j=1}^{n}a_{ij}\xi_j \qquad (i=1,2,\ldots,m)\\ \sigma_k&=\sum_{h=1}^{q}b_{kh}\eta_h \qquad (k=1,2,\ldots,p)\end{aligned}\right\}, \quad . \qquad (5a)$$

or in vector form

$$\left.\begin{aligned}r&=Ax\\ s&=By\end{aligned}\right\}, \quad . \quad . \quad . \quad . \qquad (5b)$$

introducing column vectors

$$\begin{aligned}&r=\{\rho_1,\rho_2,\ldots,\rho_m\}; \qquad x=\{\xi_1,\xi_2,\ldots,\xi_n\};\\ &s=\{\sigma_1,\sigma_2,\ldots,\sigma_p\}; \qquad y=\{\eta_1,\eta_2,\ldots,\eta_q\}.\end{aligned}$$

On account of (5a) the mp products $\rho_i\sigma_k$ are linear functions of the np

products $\xi_j\eta_h$, the matrix of the coefficients being $A \times B$, this follows at once from (4), because

$$(r \times s) = (Ax \times By) = (A \times B)(x \times y),$$

and the elements of $r \times s$ and of $x \times y$ are obviously just the products $\rho_i\sigma_k$ and $\xi_j\eta_h$.

Direct multiplication is not commutative, but we shall obtain a substitute for this property through

THEOREM I.

> *The two products* $\mathrm{A} \times \mathrm{B}$ *and* $\mathrm{B} \times \mathrm{A}$ *are related to each other by an identity*
>
> $$Q(A \times B)P^{-1} = (B \times A), \qquad . \qquad . \qquad . \qquad (6)$$
>
> *where* P *and* Q *are permutation matrices which depend only on the types of* A *and* B *and not on their elements.*

Proof.—Apart from the order the vectors $x \times y$ and $y \times x$ contain the same elements, viz. the nq products $\xi_j\eta_h$. We can therefore find a permutation matrix P of degree nq such that

$$(y \times x) = P(x \times y) \qquad . \qquad . \qquad . \qquad . \qquad (7a)$$

and similarly

$$(s \times r) = Q(r \times s), \qquad . \qquad . \qquad (7b)$$

where Q is a permutation matrix of degree mr. Evidently P and Q do not depend on the elements of x, y, u, v but only on the numbers m, n, p, q. By (5a) and (4) we have

$$(r \times s) = (A \times B)(x \times y),$$
$$(s \times r) = (B \times A)(y \times x).$$

On premultiplying the first equation by Q and substituting (7) we get

$$(s \times r) = Q(A \times B)(x \times y) = (B \times A)P(x \times y).$$

Since there is obviously no linear relation between the elements of $x \times y$, we obtain

$$Q(A \times B) = (B \times A)P$$

or

$$Q(A \times B)P^{-1} = (B \times A),$$

as was to be proved.

Let $r_1, r_2, \ldots, r_m$ be the columns of the non-singular matrix

$$R = [r_1, r_2, \ldots, r_m] = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdot & r_{2m} \\ r_{m1} & r_{m2} & \cdots & r_{mm} \end{bmatrix}, \qquad . \qquad (8)$$

i.e. suppose that the vectors

$$r_j = \{r_{1j}, r_{2j}, \ldots, r_{mj}\} \qquad (j = 1, 2, \ldots, m)$$

are linearly independent. Similarly let $s_1, s_2, \dots, s_m$ be the columns of a non-singular matrix

$$S=[s_1, s_2, \dots, s_n]=\begin{bmatrix} s_{11} & s_{12} & \cdot & s_{1n} \\ s_{21} & s_{22} & \cdot\,\cdot & s_{2n} \\ \\ s_{n1} & s_{n2} & \cdot\,\cdot & s_{nn} \end{bmatrix}, \qquad (9)$$

where

$$s_k=\{s_{1k}, s_{2k}, \dots, s_{nk}\} \qquad (k=1, 2, \dots, n).$$

We shall then prove the

LEMMA I

If $r_j(j=1, 2, \dots, m)$ *are* m *linearly independent column vectors of degree* m, *and* $s_k(k=1, 2, \dots, n)$ n *linearly independent column vectors of degree* n, *then the* mn *vectors*

$$r_j \times s_k \qquad (10)$$

of degree mn *are linearly independent.*

Proof—Supposing we had a linear relation

$$\sum_{j, k} c_{jk}(r_j \times s_k)=0,$$

or written in components

$$\sum_{j, k} c_{jk} r_{ij} s_{hk}=0 \qquad (i=1, 2, \dots, m;\ h=1, 2, \dots, m),$$

which is equivalent to the matrix equation

$$RCS'=0,$$

if we put

$$C=[c_{jk}] \qquad (j=1, 2, \dots, m;\ k=1, 2, \dots, n)$$

Since R and S are non-singular, it follows that

$$C=0,$$

i.e. the vectors (10) are linearly independent.

We shall require another lemma which goes a little farther than the preceding one:

LEMMA II

Let $r_j(j=1, 2, \dots, m)$ *and* s_k $(k=1, 2, \dots, n)$ *be defined as in Lemma I, and let* $E^{(k)}(k=1, 2, \dots, n)$ *be a set of* n *non-singular matrices of degree* m. *Then the* mn *vectors*

$$E^{(k)} r_j \times s_k \qquad (11)$$

of degree mn *are linearly independent.*

Proof.—Since R in (8) is non-singular, we may write down the matrix equation:

$$E^{(k)}R = R(R^{-1}E^{(k)}R) = RA^{(k)}, \tag{12}$$

where

$$A^{(k)} = R^{-1}E^{(k)}R = (a_{hl}^{(k)}) \tag{13}$$

is non-singular By picking out the jth column on either side of (12) we get by (8) and (13)

$$E^{(k)}r_j = \sum_h r_h a_{hj}^{(k)},$$

and after direct post-multiplication by s_k:

$$(E^{(k)}r_j \times s_k) = \sum_h a_{hj}^{(k)}(r_h \times s_k) \qquad (j=1, 2, \ldots, m, \quad k=1, 2, \ldots, n)$$

Supposing these vectors were linearly dependent, we could find mn constants z_{jk} not all zero, such that

$$\sum_{j,k} z_{jk}(E^{(k)}r_j \times s_k) = \sum_{h,j,k} z_{jk}a_{hj}^{(k)}(r_h \times s_k) = 0.$$

According to Lemma I, it now follows that

$$\sum_j a_{hj}^{(k)} z_{jk} = 0 \qquad \begin{matrix} (h=1, 2, \ldots, m) \\ (k=1, 2, \ldots, n), \end{matrix}$$

or

$$A^{(k)}z_k = 0 \qquad (k=1, 2, \ldots, m), \tag{14}$$

if we put

$$z_k = \{z_{1k}, z_{2k}, \ldots, z_{mk}\}.$$

But since $A^{(k)}$ is non-singular, we infer from (14) that

$$z_k = 0 \qquad (k=1, 2, \ldots, m),$$

i.e. the vectors (11) are linearly independent.

Now, consider the pencils

$$H = \rho A + \sigma B \quad \text{of type } m \times n \tag{15}$$

and

$$K = \rho C + \sigma D \quad \text{of type } p \times q. \tag{16}$$

From these we derive a new pencil:*

$$(H \mid K) = \rho(A \times C) + \sigma(B \times D) = \rho X + \sigma Y \tag{17}$$

of type $mp \times nq$. In the subsequent section we shall determine the minimal indices and the elementary divisors of (17), when those of (15) and (16) are known. We shall first establish some simple rules regarding $(H \mid K)$ which will facilitate the investigation.

$$(H \mid K)' = (H' \mid K'), \tag{18}$$

$$(H \mid K) \sim (K \mid H) \tag{19}$$

---

* This is, of course, not equal to $H \times K$ which, in general, will be of the second degree with respect to ρ and σ.

For by Theorem I, the products $A \times C$ and $B \times D$ can simultaneously be transformed into $C \times A$ and $D \times B$, *i.e* we have

$$Q(A \times B)P^{-1} = (B \times A),$$
$$Q(C \times D)P^{-1} = (D \times C),$$

with the same matrices Q and P. On multiplying by ρ and σ and adding we get (19)

If H^* is equivalent to H, and if K^* is equivalent to K, then

$$(H^* \mid K^*) \sim (H \mid K). \quad . \quad . \qquad (20)$$

For let

$$H^* = P_1(\rho A + \sigma B)Q_1 = \rho P_1 A Q_1 + \sigma P_1 B Q_1$$

and

$$K^* = P_2(\rho C + \sigma D)Q_2 = \rho P_2 C Q_2 + \sigma P_2 D Q_2,$$

then

$$(P_1 \times P_2)[\rho(A \times C) + \sigma(B \times D)](Q_1 \times Q_2) = \rho(P_1 A Q_1 \times P_2 C Q_2) + \sigma(P_1 B Q_1 \times P_2 D Q_2);$$

or in the notation (17):

$$(P_1 \times P_2)(H \mid K)(Q_1 \times Q_2) = (H^* \mid K^*)$$

Next, let

$$H = H_1 + H_2 = \rho(A_1 + A_2) + \sigma(B_1 + B_2)$$

By (2) we obtain

$$(H_1 \dotplus H_2 \mid K) = (H_1 \mid K) \dotplus (H_2 \mid K) \quad . \quad . \quad . \qquad (21)$$

If, on the other hand,

$$K = K_1 \dotplus K_2,$$

we have

$$(H \mid K_1 \dotplus K_2) \sim (K_1 + K_2 \mid H)$$

by (19), and hence

$$(H \mid K_1 \dotplus K_2) \sim (H \mid K_1) \dotplus (H \mid K_2). \quad . \qquad (22)$$

From (21) and (22) we derive the useful formula

$$(H_1 + H_2 \mid K_1 \dotplus K_2) \sim (H_1 \mid K_1) \dotplus (H_1 \mid K_2) \dotplus (H_2 \mid K_1) \dotplus (H_2 \mid K_2),$$

or, more generally,†

$$\left(\sum_i \cdot H_i \,\middle|\, \sum_j \cdot K_j\right) \sim \sum_{i,j} \cdot (H_i \mid K_j). \quad . \quad . \quad . \quad . \qquad (23)$$

When investigating the invariants (*i.e.* the minimal indices and elementary divisors) of the pencil we shall first of all replace H and K by their canonical forms (§ 1 (10)), which is permissible by (20). Secondly, since H and K will then appear as a direct sum, we only need to determine the invariants of the different terms

$$(H_r \mid K_s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

---

† This formula has implicitly been used before by various writers; see, *e.g.*, Roth, 1934, p. 463, Rutherford, 1933, p 435; Williamson, 1931, p. 586

by (23). Now, in the canonical form (§ 1 (10)) there occur five different kinds of pencils, viz.

$$L,\ M,\ N,\ Q \text{ and } W, \qquad . \tag{25}$$

if we for brevity leave out the indices referring to the degree and write W instead of $W(\alpha)$ or $W(\beta)$, etc Hence twenty-five different pairs (24) seem to be possible, since each pencil must be combined with itself and all the others. But on account of (19) their number at once reduces to fifteen, which may be arranged in the following scheme —

$$\begin{array}{lllll} 1\ (L \mid L) & 2.\ (L \mid M) & 3.\ (L \mid N) & 4.\ (L \mid Q) & 5\ (L \mid W) \\ & 6.\ (M \mid M) & 7.\ (M \mid N) & 8.\ (M \mid Q) & 9\ (M \mid W) \\ & & 10.\ (N \mid N) & 11.\ (N \mid Q) & 12.\ (N \mid W) \\ & & & 13.\ (Q \mid Q) & 14.\ (Q \mid W) \\ & & & & 15\ (W \mid W). \end{array} \tag{26}$$

Some of these cases may be passed over quickly by referring to symmetry in the formulæ or similar arguments. Only the last three cases have already been considered; they cover the case of non-singular pencils. For the remaining ones we shall obtain explicit solutions in the next section.

§ 4. SPECIAL PENCILS OF ZEHFUSS MATRICES.

We shall first quote Aitken's (1935) and Roth's (1934) results in our notation:

THEOREM II (*Case* 15):

If m $\geqq$ n, $\alpha \neq 0$, $\beta \neq 0$, *we have*

$$(W(\alpha) \mid W(\beta)) \sim \underset{m+n-1}{W}(\alpha\beta) \dotplus \underset{m+n-3}{W}(\alpha\beta) \dotplus \underset{m+n-5}{W}(\alpha\beta) \dotplus \ . \ . \ . \dotplus \underset{m-n+1}{W}(\alpha\beta);$$

if m < n, *we have to interchange* m *and* n *in the above result.*

THEOREM III (*Case* 14):

$$(Q_m \mid W_n(\alpha)) \sim Q_m \dotplus Q_m \dotplus \ . \ . \ . \dotplus Q_m = n \,.\, Q_m$$

for $\alpha \neq 0$.

THEOREM IV (*Case* 13).

$$(Q_m \mid Q_n) \sim 2 \,.\, Q_1 \dotplus 2 \,.\, Q_2 \dotplus \ . \ . \ . \dotplus 2 \,.\, Q_{n-1} \dotplus (m-n+1) \,.\, Q_n$$

for m $\geqq$ n. *If* m < n, *we have to interchange* m *and* n *in the result.*

An equation between, or an equivalence of, two pencils is always an identity in ρ and σ; we are therefore allowed to interchange ρ and σ. If we do this, the pencil

$$Q_k = \sigma I_k + \rho U_k' \qquad (\S\ 1\ (9))$$

becomes

$$N_k' = \rho I_k + \sigma U_k' \qquad (\S\ 1\ (3)),$$

which is equivalent to

$$N_k = \rho I_k + \sigma U_k,$$

for $N_k' = J_k N_k J_k$ (see p 51). By the same substitution

$$(Q_m \mid Q_n) = \sigma(I_m \times I_n) + \rho(U_m' \times U_n')$$

is transformed into

$$(N_m' \mid N_n') = \rho(I_m \times I_n) + \sigma(U_m' \times U_n'),$$

which is equivalent to

$$(N_m \mid N_n) = \rho(I_m \times I_n) + \sigma(U_m \times U_n) \qquad (\S\ 3\ (20))$$

Hence from Theorem IV we can at once derive the

COROLLARY (*Case* 10):

$$(N_m \mid N_n) = 2 \,.\, N_1 + 2 \,.\, N_2 + \ldots + 2 \,.\, N_{n-1} + (m - n + 1) \,.\, N_m$$

for m ≧ n. *If* m < n, *the indices* m *and* n *have to be interchanged*

Before entering into the discussion of new cases we shall explain our method by a simple example·

Find the canonical form of the pencil

$$(L_2 \mid M_3) \sim \rho(A \times C) + \sigma(B \times D) = \rho X + \sigma Y, \tag{1}$$

where

$$L_2 \sim \rho A + \sigma B; \qquad M_3 \sim \rho C + \sigma B \quad . \tag{1a}$$

and

$$X = (A \times C); \qquad Y = (B \times D) \quad . \quad . \tag{1b}$$

The pencil L_2 is of type 2 × 3, and M_3 is of type 4 × 3 (§ 1 (2) and (3)). Hence $(L_2 \mid M_3)$ is of type 8 × 9. We write (1*a*) as

$$(\rho A + \sigma B)R = R^* L_2, \quad . \quad . \quad . \tag{2}$$

$$(\rho C + \sigma D)S = S^* M_3, \quad . \quad . \quad . \tag{3}$$

where R, R^*, S, S^* are non-singular matrices. Let

$$(a)\ R = [r_1, r_2, r_3], \qquad (b)\ R^* = [r_1^*\ r_2^*], \quad . \quad . \tag{4}$$

$$(a)\ S = [s_1, s_2, s_3], \qquad (b)\ S^* = [s_0^*, s_1^*, s_2^*, s_3^*]. \quad . \tag{5}$$

The vectors r, r^*, s, s^* are therefore of degrees 3, 2, and 4 respectively. According to § 2, equations (2) and (3) are equivalent to vector chains of type L and M, which we write in the standard form (§ 2 (14))

$$r_0^* = Br_1; \qquad r_1^* = Ar_1 = Br_2; \qquad r_2^* = Ar_2 = Br_3; \qquad r_3^* = Ar_3, \quad . \tag{6}$$

$$s_0^* = Ds_1; \qquad s_1^* = Cs_1 = Ds_2; \qquad s_2 = Cs_2 = Ds_3; \qquad s_3^* = Cs_3, \quad . \tag{7}$$

where

$$r_0^* = 0; \qquad r_3^* = 0; \qquad s_0^* \neq 0; \qquad s_3^* \neq 0 \quad . \tag{8}$$

We now introduce 9 vectors of degree 9:

$$(i, j)=(r_i \times s_j) \qquad (i=1,\ 2,\ 3, \quad j=1,\ 2,\ 3) \tag{9}$$

and 16 vectors of degree 8:

$$\{h,\ k\}=(r_h^* \times s_k^*) \qquad (h=0,\ 1,\ 2,\ 3;\ k=0,\ 1,\ 2,\ 3) \tag{10}$$

The sets (9) and (10) can be arranged in two arrays:

$$(a)\ \begin{array}{ccc} (11) & (12) & (13) \\ (21) & (22) & (23), \\ (31) & (32) & (33) \end{array} \qquad (b)\ \begin{array}{cccc} \{00\} & \{01\} & \{02\} & \{03\} \\ \hline \{10\} & \{11\} & \{12\} & \{13\} \\ \{20\} & \{21\} & \{22\} & \{23\} \\ \hline \{30\} & \{31\} & \{32\} & \{33\} \end{array} \quad . \quad . \tag{11}$$

Since R and S are non-singular matrices, the 9 vectors of (11*a*) are linearly independent (§ 3, Lemma I) In the scheme (*b*) the vectors in the first and in the last row are zero in virtue of (8), the remaining 8 vectors are linearly independent according to the same lemma, because R^* and S^* are non-singular. Next, we number the diagonals of those two schemes by attaching to them the differences $i-j$ which are constant for all elements $(i,\ j)$ that lie on the same diagonal. From each diagonal we construct a matrix whose columns are the non-zero vectors which lie on this diagonal, thus:

$$(a)\ \begin{array}{l} T_{-2}=[(31)], \\ T_{-1}=[(21),\ (32)], \\ T_0=[(11),\ (22),\ (33)], \\ T_1=[(12),\ (23)], \\ T_2=[(13)]. \end{array} \qquad (b)\ \begin{array}{l} T^*_{-2}=[\{20\}], \\ T^*_{-1}=[\{10\},\ \{21\}], \\ T^*_0=[\{11\},\ \{22\}], \\ T^*_1=[\{12\},\ \{23\}], \\ T^*_2=[\{13\}]. \end{array} \quad . \quad . \tag{12}$$

Now, by (6) and (7) and by § 3 (4) we have

$$(r_h^* \times s_k^*)=(Ar_h \times Cs_k)=(A \times C)(r_h \times s_k) \qquad (h=1,\ 2,\ 3, \quad k=1,\ 2,\ 3)$$

and

$$(r_h^* \times s_k^*)=(Ar_h \times Cs_k)=(Br_{h+1} \times Ds_{k+1})=(B \times D)(r_{h+1} \times s_{k+1})$$

(for $h=0,\ 1,\ 2,\ k=0,\ 1,\ 2$); or

$$\{h,\ k\}=X(h,\ k) \qquad (h=1,\ 2,\ 3, \quad k=1,\ 2,\ 3), \tag{13}$$

$$\{h,\ k\}=Y(h+1,\ k+1) \qquad (h=0,\ 1,\ 2; \quad k=0,\ 1,\ 2) \tag{14}$$

These two equations enable us to set up vector chains for the pencil $\rho X+\sigma Y$:

$$\{00\}=Y(11);\quad \{11\}=X(11)=Y(22);\quad \{22\}=X(22)=Y(33);\quad \{33\}=X(33) \tag{15a}$$

$$\{01\}=Y(12);\quad \{12\}=X(12)=Y(23);\quad \{23\}=X(23), \quad . \quad . \quad . \quad . \tag{15b}$$

$$\{02\}=Y(13);\quad \{13\}=X(13), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{15c}$$

$$\{10\}=Y(21);\quad \{21\}=X(21)=Y(32);\quad \{32\}=X(32), \quad . \quad . \quad . \tag{15d}$$

$$\{20\}=Y(31);\quad \{31\}=X(31). \quad . \quad . \quad . \quad . \quad . \quad . \tag{15e}$$

In order to see to what submatrices these chains belong, we have to investigate their initial and final links. In (15*a*) we have by (8),

$$\{00\}=0; \qquad \{33\}=0.$$

Hence (15*a*) is of *type L* and corresponds to the submatrix L_2 In fact, we can establish the equation

$$(\rho X+\sigma Y)T_0=T_0^*L_2. \qquad (16a)$$

The chain (15*b*) starts with a zero vector and finishes with a non-zero one. It is of *type N* and equivalent to

$$(\rho X+\sigma Y)T_1=T_1^*N_2. \qquad (16b)$$

In the same way we obtain the equations

$$(\rho X+\sigma Y)T_2 \;=T_2^*N_1, \qquad (16c)$$
$$(\rho X+\sigma Y)T_{-1}=T_{-1}^*Q_2, \qquad (16d)$$
$$(\rho X+\sigma Y)T_{-2}=T_{-2}^*Q_1. \qquad (16e)$$

The classification of the chains (15) can be illustrated by means of the scheme (11) (*b*), which is divided into three areas by the horizontal lines. The middle part is occupied by non-zero vectors, while the top and the bottom are filled up by zeros. When produced, both ends of the diagonal with suffix zero enter the zero area; the corresponding chain (15*a*) therefore starts and finishes with a zero vector. The diagonals 1 and 2 only enter the zero area at the top, the diagonals −1 and −2 only at the bottom. This is characteristic for submatrices *N* and *Q* respectively. The equations (16) can be combined in

$$(\rho X+\sigma Y)T=T^*(L_2\dotplus N_2\dotplus N_1\dotplus Q_2\dotplus Q_1), \qquad (17)$$

where

$$T=[T_0,\, T_1,\, T_2,\, T_{-1},\, T_{-2}] \quad\text{and}\quad T^*=[T_0^*,\, T_1^*,\, T_2^*,\, T_{-1}^*,\, T_{-2}^*].$$

Here T has nine columns, viz. the columns of T_0, T_1, . . . taken together; they are linearly independent because they are just the vectors arranged in (11) (*a*). Hence T is a square matrix of non-zero determinant. The same can be shown for T^*, and we have therefore proved that the canonical form of $\rho X+\sigma Y$ is

$$L_2\dotplus N_2\dotplus N_1\dotplus Q_2\dotplus Q_1.$$

Turning from the example to the general case, we have two standard chains (§ 2 (14)) of lengths *m* and *n*:

$$r_0^*=Br_1;\quad r_1^*=Ar_1=Br_2;\quad r_2^*=Ar_2=Br_3;\quad \ldots;\quad r_{m-1}^*=Ar_{m-1}=Br_m;\quad r_m^*=Ar_m, \qquad (18)$$

$$s_0^*=Ds_1;\quad s_1^*=Cs_1=Ds_2;\quad s_2^*=Cs_2=Ds_3;\quad \ldots;\quad s_{n-1}^*=Cs_{n-1}=Ds_n;\quad s_n^*=Cs_n. \qquad (19)$$

As in the example, we define the vectors

$$(i, j) = (r_i \times s_j) \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n) \tag{20}$$

and

$$\{h, k\} = (r_h^* \times s_k^*) \qquad (h = 0, 1, \ldots, m;\ k = 0, 1, \ldots, n), \quad . \tag{21}$$

and arrange them in two arrays

$$(a)\ \begin{array}{cccc} (11) & (12) & & (1n), \\ (21) & (22) & \ldots & (2n), \\ & & & \\ (m1) & (m2) & \ldots & (mn), \end{array} \qquad (b)\ \begin{array}{cccc} \{00\} & \{01\} & . & \{0n\}, \\ \{10\} & \{11\} & \ldots & \{1n\}, \\ & & & \\ \{m0\} & \{m1\} & \ldots & \{mn\}. \end{array} \quad . \tag{22}$$

Again, put

$$X = (A \times C); \qquad Y = (B \times D).$$

As in (13) and (14) we have then by (18) and (19):

$$\{h, k\} = X(h, k), \qquad (h = 1, 2, \ldots, m,\ k = 1, 2, \ldots, n), \quad . \tag{23}$$

$$\{h, k\} = Y(h+1;\ k+1), \qquad (h = 0, 1, \ldots, m-1;\ k = 0, 1, \ldots, n-1), \tag{24}$$

which allows us to establish vector chains corresponding to the diagonals of the scheme (22). *E g*, if $m < n$, the chains referring to the diagonals -1, 0, and 1 are:

$$\begin{aligned} &\{10\} = Y(21);\ \ \{21\} = X(21) = Y(32);\ \ldots;\ \{m-1,\, m-2\} = X(m-1,\, m-2) \\ &\qquad = Y(m,\, m-1);\ \ \{m,\, m-1\} = X(m,\, m-1). \\ &\{00\} = Y(11);\ \ \{11\} = X(11) = Y(22);\ \ldots;\ \{m-1,\, m-1\} = X(m-1,\, m-1) \\ &\qquad = Y(m,\, m),\ \ \{m,\, m\} = X(m,\, m). \\ &\{01\} = Y(12);\ \ \{12\} = X(12) = Y(23);\ \ldots;\ \{m-1,\, m\} = X(m-1,\, m) \\ &\qquad = Y(m,\, m+1);\ \ \{m,\, m+1\} = X(m,\, m+1). \end{aligned}$$

An investigation of the initial and final terms will show us what submatrices these chains represent This can best be done by examining the corresponding diagonals -1, 0, and 1 in the scheme (22) (*b*), as will now be explained.

Let us first consider the case

$$(L_{m-1} \mid L_{n-1}).$$

In analogy to (2) and (3) we put

$$\begin{aligned} (\rho A + \sigma B)R &= R^* L_{m-1}, \\ (\rho C + \sigma D)S &= S^* L_{n-1}, \end{aligned}$$

where (*cf.* § 2 (1))

$$(a)\ R = [r_1, r_2, \ldots, r_m]; \qquad (b)\ R^* = [r_1^*, r_2^*, \ldots, r_{m-1}^*], \quad . \tag{25}$$

$$(a)\ S = [s_1, s_2, \ldots, s_n]; \qquad (b)\ S^* = [s_1^*, s_2^*, \ldots, s_{n-1}^*] \quad . \tag{26}$$

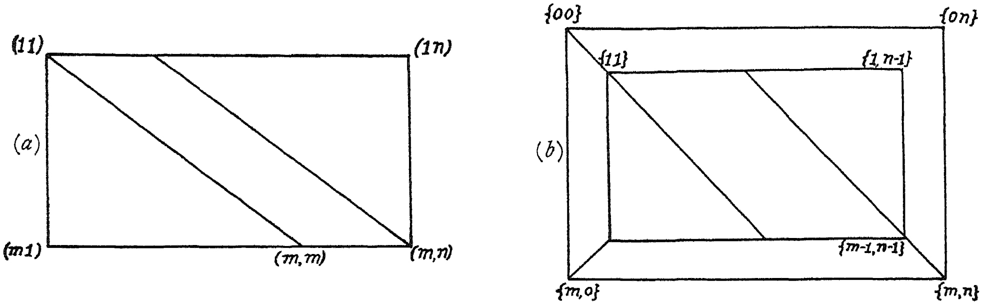
are non-singular matrices whence the column vectors r, r^*, s, s^* are of degrees m, $m-1$, n, and $n-1$ respectively. In the chains (18) and (19) we have, by § 2 (15),

$$r_0^* = 0; \qquad r_m^* = 0; \qquad s_0^* = 0; \qquad s_n^* = 0. \qquad . \quad . \quad (27)$$

Consequently, the first and last row and column of the scheme (22) (*b*) consist of zero vectors, while the remaining $(m-1)(n-1)$ vectors of degree $(m-1)(n-1)$

$$\{h, k\} \qquad (h = 1, 2, \ldots, m-1; \; k = 1, 2, \ldots, n-1)$$

are linearly independent by § 3, Lemma I Suppose that $m \leqq n$. The two schemes (22) are then adequately described by the diagrams:



(28)

From the left diagram we can find out the length of each of the $n+m-1$ diagonals. In the right-hand diagram the non-zero vectors $\{h, k\}$ fill up the inner rectangle, while the space between the two rectangles is a border of zero vectors. From (28) (*a*) we see that the diagonals

$$-(m-1), \; -(m-2), \ldots, \; -1, 0, 1, 2, \ldots, (n-m), \; (n-m+1), \\ (n-m+2), \ldots, (n-1) \qquad (29)$$

are of respective lengths:

$$1, \quad 2, \quad m-1, m, m, m, \ldots, m, \qquad m-1, \quad m-2, \ldots, \quad 1.$$

On the other hand, diagram (28) (*b*) shows us that every diagonal enters the zero area at either of its ends, *i.e.* every chain starts and finishes with a zero vector and is therefore of *type L* (see § 2). As in (12), we introduce matrices T_p and T_p^* whose columns are the vectors which lie on the *p*th diagonal of diagram (28) (*a*) and (*b*). In the latter case we have to leave out the zeros of the diagonals, if any, *e g*

$$T_{-1} = [(10), (21), \ldots, (m, m-1)]; \qquad T_{-1}^* = [\{21\}, \{32\}, \ldots, \{m-1, m-2\}]$$

The corresponding chain is of length $m-1$ (by (29)) and is therefore equivalent to

$$(\rho X + \sigma Y) T_{-1} = T_{-1}^* L_{m-2}. \qquad . \quad (30)$$

Similar equations hold for all indices Finally, we remark that

$$T=[T_{-m+1}, T_{-m+2}, \ldots, T_0, \ldots, T_{n-1}]$$

and

$$T^*=[T^*_{-m+2}, T^*_{-m+3}, \ldots, T^*_0, \ldots, T^*_{n-2}]$$

are non-singular matrices, their columns being the vectors arranged in (22) (*a*) and (*b*). We can therefore sum up equations (30) and those related to other diagonals by

$$(\rho X+\sigma Y)T=T^*\left((n-m+1)\,.\,L_{m-1}+2\,.\sum_{i=0}^{m-2}.\,L_i\right) \tag{31}$$

Special attention is to be drawn to the fact that T contains two partial matrices more than T^*, viz T_{-m+1} and T_{n-1}, because in diagram (28) (*b*) the corresponding diagonals $-m+1$ and $n-1$ consist only of zero vectors. In fact, by (27) we have

$$(\rho X+\sigma Y)(m, 1)=0,$$
$$(\rho X+\sigma Y)(1, n)=0.$$

This is accounted for by two zero columns in the canonical form, which are indicated by the two terms L_0 (see § 1 (5)) We may therefore enunciate the following theorem:—

THEOREM V (*Case* 1):

If m ≦ n, *we have*

$$(L_m \mid L_n) \sim (n-m+1)\,.\,L_m+2\,.\sum_{i=0}^{m-1}.\,L_i.$$

Since by § 3 (19)

$$(L_m \mid L_n) \sim (L_n \mid L_m),$$

the assumption $m \leqq n$ is no loss of generality. If $m > n$, we have to interchange m and n.

Again, by transposition we obtain (§ 3 (18))

$$(L'_m \mid L'_n) \sim (n-m+1)\,.\,L'_m+2\,.\sum_{i=0}^{m-1}.\,L'_i. \tag{32}$$

Since

$$L'_i=J_{i+1}M_iJ_i \tag{33}$$

(§ 1 (4)), we have $L'_i \sim M_i$, and (32) yields the

COROLLARY (*Case* 6):

$$(M_m \mid M_n) \sim (n-m+1)\,.\,M_m+2\,.\sum_{i=0}^{m-1}.\,M_i$$

for m ≦ n.

Next, consider the case

$$(L_{m-1} \mid M_n)=\rho X+\sigma Y$$

Then (18) and (19) still hold if we put

$$(\rho A+\sigma B)R=R^{*}L_{m-1},$$
$$(\rho C+\sigma D)S=S^{*}M_{n},$$

where

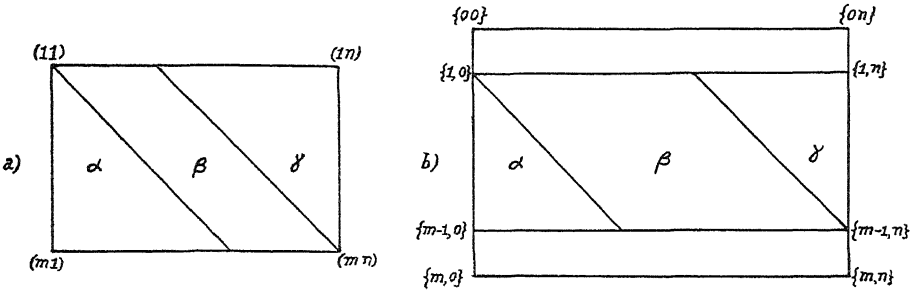
$$(a)\ R=[r_1, r_2, \ldots, r_m]; \qquad (b)\ R^{*}=[r_1^{*}, r_2^{*}, \ldots, r_{m-1}^{*}], \quad . \quad (34)$$

$$(a)\ S=[s_1, s_2, \ldots, s_n], \qquad (b)\ S^{*}=[s_0^{*}, s_1^{*}, \ldots, s_n^{*}]. \quad . \quad . \quad (35)$$

Instead of (27) we have now (§ 2 (15))

$$r_0^{*}=0; \qquad r_m^{*}=0; \qquad s_0^{*}\neq 0; \qquad s_n^{*}\neq 0. \quad . \quad . \quad . \quad (36)$$

Consequently, the first and the last row of the scheme (22) (*b*) consist of zero vectors, while the remaining vectors are a complete set† of vectors of degree $(m-1)(n+1)$ Supposing $m-1 \leqq n$, we can represent (22) (*a*) and (*b*) by the diagrams



(37)

The top and bottom story of (*b*) are occupied by zeros. We have to distinguish three categories of diagonals:

(α) The diagonals

$$-m+1, \qquad -m+2, \qquad \ldots, \qquad -1$$

are of lengths

$$1, \qquad 2, \qquad \ldots, \qquad m-1$$

The corresponding diagonals in (*b*) enter the zero area at the bottom. The chains are therefore of *type Q* (§ 2 (15)) and give rise to equations

$$(\rho X+\sigma Y)T_x=T_x^{*}Q_{x+m}, \qquad (x=-1, -2, \ldots, -m+1) \quad (38a)$$

where the columns of T_x and T_x^{*} are the vectors that lie on the xth diagonal of the schemes (37) (*a*) and (*b*).

(β) The diagonals

$$0, 1, 2, \qquad \ldots, \qquad n-m-1, \qquad n-m$$

are all of length m; in (*b*) they reach the zero border twice and the chains are therefore of *type L*. Hence

---

† *I.e.* they are linearly independent and their number equals their degree, so that a matrix whose columns [rows] they are is of non-zero determinant.

$$(\rho X+\sigma Y)T_y=T_y^*L_{m-1}. \qquad (y=0,\ 1,\ \ .\ ,\ n-m) \qquad (38\beta)$$

(γ) In the third category the diagonals

$$n-m+1, \qquad n-m+2, \qquad .\ .\ , \qquad n-2, \qquad n-1$$

are of lengths

$$m-1, \qquad m-2, \qquad .\ \ , \qquad 2, \qquad 1.$$

In (*b*) they enter the zero area at the top, the corresponding chains are therefore of *type N:*

$$(\rho X+\sigma Y)T_z=T_z^*N_{n-z}. \qquad (z=n-m+1,\ n-m+2,\ .\ .\ ,\ n-1) \qquad (38\gamma)$$

We can summarise the three equations (38) in

$$(\rho X+\sigma Y)T=T^*\left(\sum_{i=1}^{m-1}.\ Q_i\dotplus(n-m+1)\quad L_{m-1}+\sum_{i=1}^{m-1}.\ N_i\right), \qquad . \qquad (39)$$

where T and T^* consist of the columns of T_x, T_y, T_z and T_x^*, T_y^*, T_z^* taken together; as before, we see that they are non-singular matrices whose columns are the complete sets of vectors arranged in diagram (37) (*b*). It can easily be shown that the formula still holds when $m-1=n$, in which case area (β) does not appear Replacing $m-1$ by m we have·

THEOREM VI (*Case* 2):

$$(L_m \mid M_n)\sim(n-m)\ .\ L_m\dotplus\sum_{i=1}^{m}.\ Q_i\dotplus\sum_{i=1}^{m}.\ N_i \qquad . \qquad . \qquad (40)$$

for m $\leqq$ n *and*

$$(L_m \mid M_n)\sim(m-n)\ .\ M_n\dotplus\sum_{i=1}^{n}.\ Q_i\dotplus\sum_{i=1}^{n}.\ N_i$$

for m $\geqq$ n

The second part of the theorem easily follows from the first one; for by transposition we obtain from (40):

$$(L_m' \mid M_n')\sim(n-m)\ .\ L_m'+\sum_{i-1}^{m}.\ Q_i'\dotplus\sum_{i-1}^{m}.\ N_i.$$

By (33) we have

$$L_m'\sim M_m; \qquad M_n'\sim L_n, \qquad . \qquad . \qquad . \qquad . \qquad (41)$$

and similarly

$$Q_i'=J_iQ_iJ_i; \qquad \text{i.e. } Q_i'\sim Q_i, \qquad . \qquad . \qquad . \qquad . \qquad (42)$$

$$N_i'=J_iN_iJ_i; \qquad \text{i.e. } N_i'\sim N_i \qquad . \qquad . \qquad . \qquad (43)$$

Substituting this and using § 3 (19) we get

$$(L_n \mid M_m)\sim(n-m)\ .\ M_m\dotplus\sum_{i-1}^{m}.\ Q_i\dotplus\sum_{i=1}^{m}.\ N_i, \qquad (m\leqq n)$$

which is identical with the second statement of the above theorem when

m and n are interchanged. The case $m = n$ is especially interesting, for although both L_n and M_n are singular pencils, the composite pencil $(L_n \mid M_n)$ is *non-singular* of degree $n(n+1)$, its determinant being equal to

$$(\rho\sigma)^{\frac{1}{2}n(n+1)}.$$

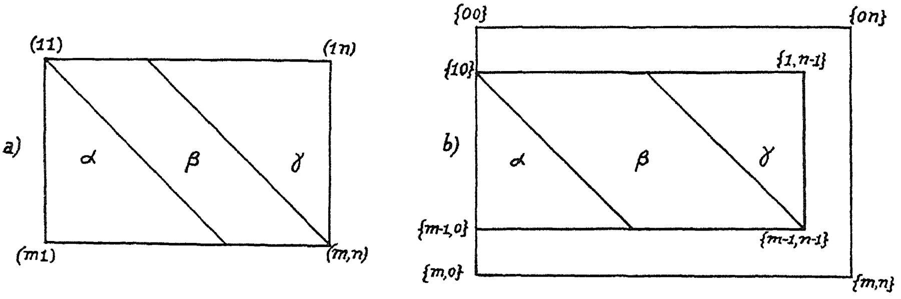
We now proceed to the discussion of the pencil

$$(L_{m-1} \mid Q_n) = \rho X + \sigma Y$$

Instead of (36) we have here

$$r_0^* = 0; \qquad r_m^* = 0, \qquad s_0^* \neq 0, \qquad s_n^* = 0 \qquad (\S\ 2\ (15))\ . \qquad (44)$$

First let $m \leq n$ The characteristic diagrams which represent (22) (*a*) and (*b*) are in this case



(45)

where the space between the two rectangles in (*b*) is occupied by zero vectors in virtue of (44). Again, we have three categories of diagonals which may be arranged in the following table containing the number of each diagonal, its length, and the submatrix which it represents according to § 2 (15):—

	α	β	γ
Diagonal	$-1, \quad -2, \quad \ldots, \quad -m+1$	$0, \quad 1, \quad \ldots, n-m$	$n-m+1, n-m+2, \ldots, n-1$
Length .	$m-1, m-2, \ldots, \quad 1$	$m, \quad m, \quad \ldots, \quad m$	$m-1, \quad m-2, \quad \ldots, \quad 1$
Submatrix	$Q_{m-1}, Q_{m-2}, \ldots, \quad Q_1$	$L_{m-1}, L_{m-1}, \ldots, L_{m-1}$	$L_{m-2}, \quad L_{m-3}, \quad \ldots, L_0$

Hence we can establish three kinds of equations:

$$(\rho X + \sigma Y)T_x = T_x^* Q_{m+x} \qquad (x = -1, -2, \ldots, -m+1) \quad . \quad (46\alpha)$$

$$(\rho X + \sigma Y)T_y = T_y^* L_{m-1} \qquad (y = 0, 1, \ldots, n-m) \qquad (46\beta)$$

$$(\rho X + \sigma Y)T_z = T_z^* L_{n-z-1} \qquad (z = n-m+1, \ldots, n-1), \qquad (46\gamma)$$

or

$$(\rho X+\sigma Y)T=T^*\left(\sum_{i=1}^{m-1}.\,Q_i\dotplus(n-m+1)\,.\,L_{m-1}\dotplus\sum_{i=0}^{m-2}L_i\right),\quad (m\leqq n)\quad (47)$$

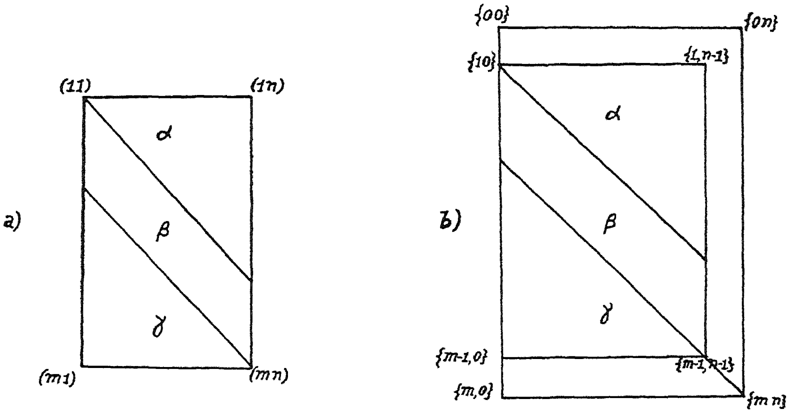
where T and T^* are non-singular matrices whose columns are those of T_x, T_y, T_z taken together, *i e.* the vectors

$$(i, j)\qquad (i=1, 2, \ldots, m;\ j=1, 2, \ldots, n)$$

and

$$\{h, k\},\qquad (h=1, 2, \ldots, m-1;\ k=0, 1, \ldots, n-1)$$

which again form two complete sets of vectors If $m \geqq n$, the characteristic rectangles assume the shapes.



(48)

and the table of diagonals becomes:

	a	β	γ
Diagonal .	$0, \quad 1, \quad \ldots, n-1$	$-1, -2, \quad . \quad , -m+n$	$-m+n-1, \; -m+n-2, \ldots, \; -m+1$
Length .	$n, \quad n-1, \ldots, \quad 1$	$n, \quad n, \quad \ldots, \quad n$	$n-1, \quad n-2, \quad \ldots, \quad 1$
Submatrix	$L_{n-1}, L_{n-2}, \ldots, L_0$	$Q_n, Q_n, \ldots, Q_n$	$Q_{n-1}, \quad Q_{n-2}, \quad \ldots, \quad Q_1$

In the same manner we now get

$$(\rho X+\sigma Y)T=T^*\left(\sum_{i=0}^{n-1}.\,L_i\dotplus(m-n)\,.\,Q_n\dotplus\sum_{i=1}^{n-1}.\,Q_i\right).\quad (m\geqq n)\ .\quad (49)$$

It is easy to see that (47) and (49) yield identical results when $m=n$.

THEOREM VII (*Case* 4):

$$(L_{m-1} \mid Q_n) \sim (n-m) \,.\, L_{m-1} + \sum_{i=1}^{m-1} \cdot Q_i \dotplus \sum_{i=0}^{m-1} . L_i, \quad \text{for } m \leqq n,$$

$$(L_{m-1} \mid Q_n) \sim (m-n) \,.\, Q_n + \sum_{i=1}^{n-1} \cdot Q_i + \sum_{i=0}^{n-1} L_i, \quad \text{for } m \geqq n$$

By transposition and by (41) and (42) we obtain the

COROLLARY I (*Case* 8)·

$$(M_{m-1} \mid Q_n) \sim (n-m) \,.\, M_{m-1} + \sum_{i=1}^{m-1} \cdot Q_i + \sum_{i=0}^{m-1} . M_i, \quad \text{for } m \leqq n,$$

$$(M_{m-1} \mid Q_n) \sim (m-n) \,.\, Q_n + \sum_{i=1}^{n-1} \cdot Q_i + \sum_{i=0}^{n-1} . M_i, \quad \text{for } m \geqq n.$$

Further corollaries can be derived by interchanging ρ and σ in the above results. This process replaces Q_i and N_i and *vice versa*, while L_i and M_i are transformed into

$$J_i L_i J_{i+1} \quad \text{and} \quad J_{i+1} M_i J_i$$

and therefore remain equivalent to themselves

COROLLARY II (*Case* 3):

$$(L_{m-1} \mid N_n) \sim (n-m) \,.\, L_{m-1} \dotplus \sum_{i=1}^{m-1} . N_i \dotplus \sum_{i=0}^{m-1} . L_i, \quad \text{for } m \leqq n,$$

$$(L_{m-1} \mid N_n) \sim (m-n) \,.\, N_n \dotplus \sum_{i=1}^{n-1} . N_i \dotplus \sum_{i=0}^{n-1} . L_i, \quad \text{for } m \geqq n.$$

COROLLARY III (*Case* 7)·

$$(M_{m-1} \mid N_n) \sim (n-m) \,.\, M_{m-1} + \sum_{i=1}^{m-1} . N_i \dotplus \sum_{i=0}^{m-1} . M_i, \quad \text{for } m \leqq n,$$

$$(M_{m-1} \mid N_n) \sim (m-n) \,.\, N_n + \sum_{i=1}^{n-1} . N_i \dotplus \sum_{i=0}^{m-1} . M_i, \quad \text{for } m \geqq n.$$

Next, consider the pencil

$$(N_m \mid Q_n) = \rho X + \sigma Y.$$

We have now to put

$$(\rho A + \sigma B) R = R^* N_m,$$
$$(\rho C + \sigma D) S = S^* Q_n,$$

where

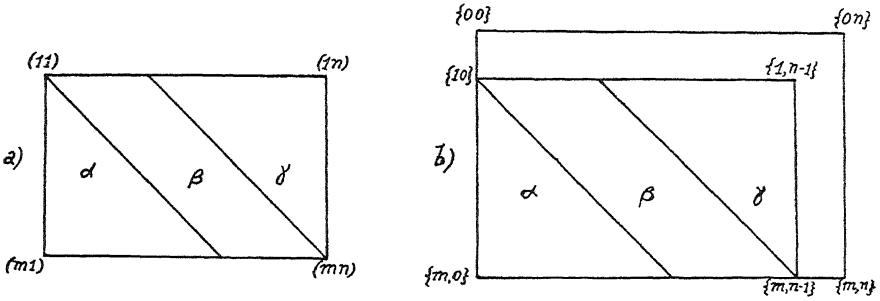
$$(a)\ R = [r_1, r_2, \ldots, r_m]; \qquad (b)\ R^* = [r_1^*, r_2^*, \ldots, r_m^*]; \qquad (50)$$

$$(a)\ S = [s_1, s_2, \ldots, s_n]; \qquad (b)\ S^* = [s_0^*, s_1^*, \ldots, s_{n-1}^*], \quad . \qquad (51)$$

are non-singular matrices, and

$$r_0^* = 0; \qquad r_m^* \neq 0; \qquad s_0^* \neq 0; \qquad s_n^* = 0. \quad . \quad . \quad . \qquad (52)$$

Supposing that $m \leqq n$, we can illustrate the different vector chains by the diagrams:



(53)

from which we tabulate the diagonals as follows:—

	α	β	γ
Diagonal .	$-1,\ -2,\ .\ .\ ,\ -m+1$	$0,\ 1,\ .\ .\ ,\ n-m-1$	$n-m,\ n-m+1,\ \ ,\ n-1$
Length .	$m-1,\ m-2,\ .\ .\ .\ ,\ 1$	$m,\ m,\ .\ .\ ,\ m$	$m,\ m-1,\ .\ .\ ,\ 1$
Submatrix .	$M_{m-1},\ M_{m-2},\ .\ .\ .,\ M_1$	$N_m,\ N_m,\ .\ .\ ,\ N_m$	$L_{m-1},\ L_{m-2},\ .\ .\ .,\ L_0$

For the diagonals of the area α do not enter the zero border at all, those of β enter it once, namely at the top, and those of γ twice. Corresponding to these three kinds of diagonals we have three sets of equations:

$$(\rho X+\sigma Y)T_x=T_x^*M_{m+x} \qquad (x=-1,\ -2,\ .\ .\ .,\ -m+1)\ . \quad .\ (54\alpha)$$

$$(\rho X+\sigma Y)T_y=T_y^*N_m \qquad (y=0,\ 1,\ .\ .\ ,\ n-m-1) \quad .\ \quad .\ (54\beta)$$

$$(\rho X+\sigma Y)T_z=T_z^*L_{n-z-1} \qquad (z=n-m,\ n-m+1,\ .\ .\ .,\ n-1).\ .\ (54\gamma)$$

Let T be the matrix whose columns are those of T_x, T_y, T_z taken together; it will then evidently possess as columns the mn vectors (i, j) arranged in (53) (a); they form a complete set of vectors and hence T is square and non-singular. If we define T^* in a similar way, we have by (54, α, β, γ):

$$(\rho X+\sigma Y)T=T^*\left(\sum_{i=1}^{m-1}.\ M_i+(n-m)\ .\ N_m+\sum_{i=0}^{m-1}.\ L_i\right)=T^*(\rho X_1+\sigma Y_1). \qquad (55)$$

But T^* is not a square matrix, in that it contains only $nm-1$ columns, viz. all non-zero vectors of the scheme (53) (b) except $\{m, 0\}=(r_m^* \times s_0^*)\neq 0$, which lies on the diagonal with suffix $-m$. This diagonal has not been considered since it does not occur in diagram (53) (a). The matrix

$$T_1^*=[\{m, 0\},\ T^*]$$

is, however, square and non-singular, and we shall write the right-hand side of (55) as

$$[\{m, 0\}, T^*] \begin{bmatrix} \cdot \\ \rho X_1 + \sigma Y_1 \end{bmatrix},$$

i.e. the canonical form is to be augmented by a zero row which we denote by M_0 (§ 1 (5*b*)). This is analogous to the term L_0, which is due to the fact that a non-zero diagonal with index $n-1$ occurs in (*a*) but not in (*b*). We have therefore proved

THEOREM VIII (*Case* 11).

$$(N_m \mid Q_n) \sim \sum_{i=0}^{m-1} . M_i \dotplus (n-m) \quad N_m + \sum_{i=0}^{m-1} . L_i, \quad \text{for } m \leqq n,$$

$$(N_m \mid Q_n) \sim \sum_{i=0}^{n-1} . M_i \dotplus (m-n) . Q_n \dotplus \sum_{i=0}^{n-1} . L_i, \quad \text{for } m \geqq n.$$

The second result easily follows from the first one by interchanging ρ and σ and m and n (see p. 72). Although generated by two non-singular pencils N_m and Q_n, the pencil $(N_m \mid Q_n)$ is always singular.

For the remaining cases our method has to be modified owing to the appearance of the submatrix $W_m(\alpha)$, $\alpha \neq 0$.

We shall first consider the pencil

$$(W_m(\alpha) \mid L_n) = \rho X + \sigma Y.$$

By § 1 (8) we have

$$W_m(\alpha) = \rho(\alpha I_m + U'_m) + \sigma I_m. \quad (\alpha \neq 0) \qquad . \quad . \quad (56)$$

Put

$$\rho C + \sigma D = L_n \qquad . \quad . \quad (57)$$

or

$$(\rho C + \sigma D) S = S^* L_n, \qquad . \quad . \quad . \quad (58)$$

where

$$(a)\ S = [s_0, s_1, \ldots, s_n], \qquad (b)\ S^* = [s_1^*, s_2^*, \ldots, s_n^*] \qquad . \quad (59)$$

are square matrices of non-zero determinants; they are, in fact, unit matrices of degree $n+1$ and n. We have then by § 3 (17):

$$X = (E \times C); \qquad Y = (I \times D), \qquad . \quad (60)$$

where

$$E = \alpha I_m + U'_m \quad (\alpha \neq 0) \qquad . \quad . \quad . \quad (61)$$

is a non-singular matrix; for brevity we shall write I instead of I_m. According to § 2, equation (58) above is equivalent to a vector chain of *type* L, namely,

$$0 = Ds_0; \quad s_1^* = Cs_0 = Ds_1; \quad s_2^* = Cs_1 = Ds_2; \quad \ldots; \quad s_n^* = Cs_{n-1} = Ds_n; \quad Cs_n = 0. \qquad (62)$$

Next, let

$$r_1, r_2, \ldots, r_m$$

be any set of m linearly independent column vectors of degree m. By § 3, Lemma II, the $m(n+1)$ vectors

$$(i,j)=(E^j r_i \times s_j) \quad (i=1,2,\ldots,m,\ j=0,1,\ldots,n) \tag{63}$$

each of which has $m(n+1)$ elements, are linearly independent and so are the vectors

$$\{h,k\}=(E^k r_h \times s_k^*) \quad (h=1,2,\ldots,m;\ k=1,2,\ldots,n), \tag{64}$$

which are of degree mn. We arrange these two complete sets in two schemes:

$$(a)\ \begin{array}{cccc} (10) & (11) & \ldots & (1n) \\ (20) & (21) & \ldots & (2n) \\ & & & \\ (m0) & (m1) & \ldots & (mn) \end{array} \qquad (b)\ \begin{array}{cccc} \{11\} & \{12\} & \ldots & \{1n\} \\ \{21\} & \{22\} & \ldots & \{2n\} \\ & & & \\ \{m1\} & \{m2\} & \ldots & \{mn\}. \end{array} \tag{65}$$

By (60) and (62) we have

$$Y(i,0)=0; \qquad X(i,n)=0 \qquad (i=1,2,\ldots,m), \tag{66}$$

$$X(i,j-1)=(E^j r_i \times Cs_{j-1})=(E^j r_i \times Ds_j)=Y(i,j)=\{i,j\} \tag{67}$$

(for $i=1,2,\ldots,m$; $j=1,2,\ldots,n$). These relations can be summarised in the vector chains.

$$0=Y(i,0);\quad \{i,1\}=X(i,0)=Y(i,1);\quad \ldots;\quad \{i,n\}=X(i,n-1)=Y(i,n);\quad X(i,n)=0 \tag{68}$$

(for $i=1,2,\ldots,m$). If we put

$$T_i=[(i0),(i1),\ldots,(in)]; \qquad T_i^*=[\{i1\},\{i2\},\ldots,\{in\}], \tag{69}$$

equation (68) is equivalent to (see § 2 (1, a) and (1, b))

$$(\rho X+\sigma Y)T_i=T_i^* L_n \quad (i=1,2,\ldots,m)$$

and

$$(\rho X+\sigma Y)T=T^*(L_n \dotplus L_n \dotplus \ldots \dotplus L_n)=T^*(m\,.\,L_n), \tag{70}$$

where

$$T=[T_1,T_2,\ldots,T_m]; \qquad T^*=[T_1^*,T_2^*,\ldots,T_m^*]$$

Here T and T^* are square matrices of non-zero determinants because their columns are just the vectors tabulated in (65) (a) and (b). We have therefore proved:

THEOREM IX (*Case* 5):

If $\alpha \neq 0$,

$$(W_m(\alpha) \mid L_n) \sim m\,.\,L_n.$$

By transposition we obtain the

COROLLARY (*Case* 9):

If $\alpha \neq 0$,

$$(W_m(\alpha) \mid M_n) \sim m\,.\,M_n.$$

For $L_n' \sim M_n$ by (41), and $W_m'(\alpha)=J_m W_m(\alpha) J_m$.

We can treat the case

$$(W_m(\alpha) \mid N_n) = \rho X + \sigma Y$$

briefly, as it is very similar to the preceding one. Putting

$$(\rho C + \sigma D) S = S^* N_n,$$

we now have the chain

$$0 = D s_1; \quad s_1^* = C s_1 = D s_2; \quad . \quad ., \quad s_{n-1}^* = C s_{n-1} = D s_n, \quad s_n^* = C s_n \neq 0, \qquad (71)$$

where

$$S = [s_1, s_2, \ldots, s_n] \quad \text{and} \quad S^* = [s_1^*, s_2^*, \ldots, s_n^*]$$

are non-singular and square. Again, let

$$(i, j) = (E^j r_i \times s_j) \qquad (i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n) \qquad (72)$$

and

$$\{h, k\} = (E^k r_h \times s_k^*) \quad (h = 1, 2, \ldots, m; \; k = 1, 2, \ldots, n). \qquad (73)$$

It then follows that

$$Y(i, 1) = 0; \qquad X(i, j) = Y(i, j+1); \qquad \ldots; \qquad X(i, n) \neq 0, \qquad (74)$$

which yields the vector chains

$$0 = Y(i, 1), \; \{i, 1\} = X(i, 1) = Y(i, 2); \; \ldots; \; \{i, n-1\} = X(i, n-1) = Y(i, n); \; X(i, n) \neq 0 \quad (7$$

(for $i = 1, 2, \ldots, m$), *i e.* we have m vector chains of *type N* each of which represents a submatrix N_n; the set of equation (75) is therefore equivalent to

$$(\rho X + \sigma Y) T_i = T_i^* N_n, \quad (i = 1, 2, \ldots, m), \qquad (76)$$

where

$$T_i = [(i, 1), (i, 2), \ldots, (i, n)]; \qquad T_i^* = [\{i, 1\}, \{i, 2\}, \ldots, \{i, n\}].$$

Hence

$$(\rho X + \sigma Y) T = T^* (N_n + N_n + \ldots \dotplus N_n) = T^* (m \;\; N_n).$$

As before, we can show that T and T^* are square matrices of non-zero determinants, and we have thus proved:

THEOREM X (*Case* 12):

If $\alpha \neq 0$,

$$(W_m(\alpha) \mid N_n) \sim m \, . \, N_n.$$

This is the last of the fifteen pencils (§ 3 (26)) whose canonical forms we wished to determine. Hence we have obtained a method of finding the canonical form of any pencil of Zehfuss matrices, since the most general such pencil can be transformed into an aggregate of the fifteen pencils which we have considered

§ 5. PENCILS OF COMPOUND MATRICES.

If A is a matrix of type $m \times n$, its pth compound $A^{(p)}$ ($p \leqq m$; $p \leqq n$) is a matrix whose elements are the

$$\binom{m}{p}\binom{n}{p}$$

minors of degree p that can be formed from the elements of A. The general element of $A^{(p)}$ will be denoted by (see Aitken, 1935, p. 365):

$$\binom{i_1, i_2, \ldots, i_p}{j_1, j_2, \ldots, j_p} = \binom{i^{(p)}}{j^{(p)}} = \begin{vmatrix} a_{i_1 j_1} & \cdot & a_{i_1 j_p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{i_p j_1} & \ldots & a_{i_p j_p} \end{vmatrix}.$$

The two groups of indices $i^{(p)}$ and $j^{(p)}$ are arranged in *dictionary order*, and the $\binom{m}{p}$ combinations of indices $i^{(p)}$ refer to the rows of $A^{(p)}$, while the $\binom{n}{p}$ combinations $j^{(p)}$ specify the columns. As is well known, compound matrices obey the multiplicative law

$$(AB)^{(p)} = A^{(p)}B^{(p)}. \qquad (1)$$

We also note the rule

$$(A')^{(p)} = (A^{(p)})'. \qquad (2)$$

Again, if A is of non-zero determinant, then so is $A^{(p)}$. This follows from (1) by putting $B = A^{-1}$.

We shall now consider the pth compound of a direct sum. Let

$$m = m_1 + m_2,$$
$$n = n_1 + n_2,$$

and

$$A = B \dotplus C = \begin{bmatrix} B & \cdot \\ \cdot & C \end{bmatrix},$$

where B is of type $m_1 \times n_1$, and C of type $m_2 \times n_2$. We can then prove the

THEOREM XI (Littlewood, 1935, p. 375):

Two permutation matrices P *and* R *can be found such that*

$$P(B \dotplus C)^{(p)}R = \sum_{s=0}^{p} \cdot (B^{(p-s)} \times C^{(s)}), \qquad (3)$$

where P *and* R *depend only on the number of rows and columns in* B *and* C *and not on their elements.*

In (3) we have to put $B^{(0)} = C^{(0)} = 1$ and to omit every term that is meaningless, *i.e.* in which the upper index exceeds the number of either rows or columns. Littlewood (*loc. cit.*) proves the theorem for the case of a square matrix; his arguments hold equally for a rectangular matrix, and would show that P and R do not depend on the elements of this matrix. But we shall here deduce the theorem directly from the definition of a compound matrix. According to the partition of A we shall denote the elements by

$$\begin{pmatrix} i_1, i_2, \dots, i_{p-s}, k_1, k_2, \dots, k_s \\ j_1, j_2, \dots, j_{p-t}, h_1, h_2, \dots, h_t \end{pmatrix} = \begin{pmatrix} i^{(p-s)} k^{(s)} \\ j^{(p-t)} h^{(t)} \end{pmatrix}, \qquad (s = 0, 1, 2, \dots, p) \qquad (4)$$

where

$$1 \leqq i_1 < i_2 < \dots \leqq m_1; \quad 1 \leqq j_1 < j_2 < \dots \leqq n_1, \qquad (5)$$

$$1 \leqq k_1 < k_2 < \dots \leqq m_2; \quad 1 \leqq h_1 < h_2 < \dots \leqq n_2. \qquad (6)$$

First of all it is easy to see that an element is zero unless $s = t$ (Aitken, 1935, p. 366), and hence that every non-zero minor of A is a product of a minor of B and a minor of C (one of which must in the limiting cases $p = s$ and $s = 0$ be replaced by unity). In fact, we have

$$\begin{pmatrix} i^{(p-s)} k^{(s)} \\ j^{(p-s)} h^{(s)} \end{pmatrix} = \begin{pmatrix} i^{(p-s)} \\ j^{(p-s)} \end{pmatrix} \begin{pmatrix} k^{(s)} \\ h^{(s)} \end{pmatrix} \qquad (7)$$

Next, we pick out all combinations of indices $i^{(p-s)}$, $j^{(p-s)}$, $k^{(s)}$, $h^{(s)}$ which are possible for a fixed value of s and arrange them in each of these four groups in dictionary order. By (7) the corresponding elements form the matrix

$$B^{(p-s)} \times C^{(s)}. \qquad (8)$$

This is to be done for $s = 0, 1, 2, \dots, p$. Since the submatrices belonging to different values of s are evidently isolated, we obtain

$$\sum_{i=0}^{p} \cdot (B^{(p-s)} \times C^{(s)}). \qquad (9)$$

The final arrangement of *all* indices is, of course, not the dictionary order. But we may say that (9) is derived from $A^{(p)}$ by a permutation of rows and columns which depends only on the partition of A into B and C. This proves the theorem.

In this paragraph we shall deal with pencils of the form

$$(H; p) = \rho A^{(p)} + \sigma B^{(p)} \qquad (10)$$

when the invariants of

$$H = \rho A + \sigma B \qquad (11)$$

are known. We may assume that (11) is already in canonical form, because if

$$S(\rho A + \sigma B)T = \rho A_1 + \sigma B_1,$$

i.e.

$$SAT = A_1; \quad SBT = B_1,$$

we obtain by (1)

$$S^{(p)}(\rho A^{(p)}+\sigma B^{(p)})T^{(p)}=\rho A_1^{(p)}+\sigma B_1^{(p)}.$$

Again, if the given pencil can be written as a direct sum

$$\rho A+\sigma B=\rho(A_1\dotplus A_2)+\sigma(B_1\dotplus B_2),$$

we have by Theorem XI:

$$PA^{(p)}R=\sum_{s=0}^{p}\cdot(A_1^{(p-s)}\times A_2^{(s)}),$$

$$PB^{(p)}R=\sum_{s=0}^{p}\cdot(B_1^{(p-s)}\times B_2^{(s)}),$$

with the same matrices P and R for A and B. Hence, on multiplying by ρ and σ and adding, we get

$$P(\rho A^{(p)}+\sigma B^{(p)})R=\sum_{s=0}^{p}\cdot\{\rho(A_1^{(p-s)}\times A_2^{(s)})+\sigma(B_1^{(p-s)}\times B_2^{(s)})\};$$

or in the notation of § 3 (17):

$$\rho A^{(p)}+\sigma B^{(p)}\sim\sum\cdot(\rho A_1^{(p-s)}+\sigma B_1^{(p-s)}\mid\rho A_2^{(s)}+\sigma B_2^{(s)}).$$

We may assume that $\rho A_1+\sigma B_1$ is one of the elementary submatrices (§ 1), and since the case of direct products was fully treated in the last section, we have only to find the canonical form of

$$(H;\ p)=\rho A^{(p)}+\sigma B^{(p)}$$

when $H=\rho A+\sigma B$ is one of the elementary pencils

$$L_n,\ M_n,\ N_n,\ Q_n,\ W_n(\alpha)$$

(§ 1 (2), (3), (4), and (8)). The last two of these pencils also occur in the canonical forms of a non-singular pencil and have been treated by Aitken (1935) and Littlewood (1935). We state their results in our notation·

THEOREM XII (Littlewood, 1935, Theorem IV):

If $\alpha\neq 0$,

$$(W_n(\alpha);\ p)\sim\sum_{s=0}^{\left[\frac{q-1}{2}\right]}\cdot c_s\cdot W_{q-2s}(\alpha^p),$$

where $q=p(n-p)+1$, *and where* c_s *is the number of partitions of* s *minus the number of partitions of* $s-1$ *into* p *parts each* $\leqq n-p$; $\left[\frac{q-1}{2}\right]$, *as usual, denotes the greatest integer less that or equal to* $\frac{q-1}{2}$.

THEOREM XIII (Littlewood, 1935, Theorem V): †

$$(Q_n;\ p)\sim\sum_{s=p}^{n}\cdot\binom{s-2}{p-2}\cdot Q_{n+1-s}.$$

---

† Littlewood denotes the coefficient of $Q_{n-p+1-i}$ by c_i and defines c_i as the number of partitions into i parts each $\leqslant m-1$. But it is easy to see that this number is equal to $\binom{i+p-2}{i}$.

If we interchange ρ and σ in Theorem XIII and transpose the matrices, we obtain the

COROLLARY:

$$(N_p;\ p) \sim \sum_{s=p}^{n} \cdot \binom{s-2}{p-2} \cdot N_{n+1-s}.$$

We now turn to the discussion of $\rho A^{(p)} + B^{(p)}$ when

$$\rho A + \sigma B \sim L_n.$$

It will be useful first to consider a simple example. Let $p=3$ and

$$(\rho A + \sigma B)R = R^* L_4, \tag{12}$$

where

$$R = [r_0, r_1, r_2, r_3, r_4] \quad \text{and} \quad R^* = [r_1^*, r_2^*, r_3^*, r_4^*] \tag{13}$$

are non-singular matrices (*e.g.* unit matrices) According to § 2, equation (12) above is equivalent to a vector chain of *type L*, viz.:

$$0 = Br_0, \quad Ar_0 = Br_1 = r_1^*; \quad Ar_1 = Br_2 = r_2^*, \quad Ar_2 = Br_3 = r_3^*; \quad Ar_3 = Br_4 = r_4^*; \quad Ar_4 = 0. \tag{14}$$

Put

$$(i, k, h) = [r_i, r_k, r_h]^{(3)}, \qquad (0 \leqq i < k < h \leqq 4). \tag{15}$$

These are $\binom{5}{3}$ column vectors of degree $\binom{5}{3}$. They are linearly independent, because by definition they are the columns of $R^{(3)}$. Similarly, we define the $\binom{4}{3}$ vectors

$$\{i, k, h\} = [r_i^*, r_k^*, r_h^*]^{(3)}, \qquad (1 \leqq i < k < h \leqq 4) \tag{16}$$

of degree $\binom{4}{3}$, which are likewise linearly independent, being the columns of $(R^*)^{(3)}$ By (14) we have

$$A[r_{i-1}, r_{k-1}, r_{h-1}] = B[r_i, r_k, r_h] = [r_i^*, r_k^*, r_h^*]$$

(for $1 \leqq i < k < h \leqq 4$), or taking the third compound we obtain:

$$A^{(3)}(i-1, k-1, h-1) = B^{(3)}(i, k, h) = \{i, k, h\} \tag{17}$$

Again,

$$B^{(3)}(0, k, h) = 0; \qquad A^{(3)}(i, k, 4) = 0. \tag{18}$$

Equations (17) and (18) enable us to set up the following vector chains for the pencil $\rho A^{(3)} + \sigma B^{(3)}$:—

$$B^{(3)}(0, 1, 2) = 0; \quad A^{(3)}(0, 1, 2) = B^{(3)}(1, 2, 3) = \{1, 2, 3\}; \quad A^{(3)}(1, 2, 3) = B^{(3)}(2, 3, 4) = \{2, 3, 4\}; \quad A^{(3)}(2, 3, 4) = 0, \tag{19a}$$

$$B^{(3)}(0, 1, 3) = 0; \quad A^{(3)}(0, 1, 3) = B^{(3)}(1, 2, 4) = \{1, 2, 4\}; \quad A^{(3)}(1, 2, 4) = 0, \tag{19b}$$

$$B^{(3)}(0, 1, 4) = 0; \quad A^{(3)}(0, 1, 4) = 0, \tag{19c}$$

$$B^{(3)}(0, 2, 3)=0; \quad A^{(3)}(0, 2, 3)=B^{(3)}(1, 3, 4)=\{1, 3, 4\}; \quad A^{(3)}(1, 3, 4)=0, \qquad (19d)$$

$$B^{(3)}(0, 2, 4)=0; \quad A^{(3)}(0, 2, 4)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19e)$$

$$B^{(3)}(0, 3, 4)=0; \quad A^{(3)}(0, 3, 4)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19f)$$

These six chains are all of *type* L (§ 2), and correspond to terms L_2, L_1, L_0, L_1, L_0, L_0 in the canonical form, where L_0 indicates that the canonical form contains a zero column due to a constant vector which annihilates all matrices of the pencil. Now, let

$$\begin{aligned}
T_{12}&=[(0, 1, 2), \quad (1, 2, 3), \quad (2, 3, 4)], & T^*_{12}&=[\{1, 2, 3\}, \quad \{2, 3, 4\}],\\
T_{13}&=[(0, 1, 3), \quad (1, 2, 4)], & T^*_{13}&=[\{1, 2, 4\}],\\
T_{14}&=[(0, 1, 4)], & &\\
T_{23}&=[(0, 2, 3), \quad (1, 3, 4)], & T^*_{23}&=[\{1, 3, 4\}],\\
T_{24}&=[(0, 2, 4)], & &\\
T_{34}&=[(0, 3, 4)]. & &
\end{aligned}$$

Equations (19) are equivalent to

$$\begin{aligned}
&(\rho A^{(3)}+\sigma B^{(3)})T_{12}=T^*_{12}L_2,\\
&(\rho A^{(3)}+\sigma B^{(3)})T_{13}=T^*_{13}L_1,\\
&(\rho A^{(3)}+\sigma B^{(3)})T_{23}=T^*_{23}L_1,\\
&(\rho A^{(3)}+\sigma B^{(3)})T_{14}=(\rho A^{(3)}+\sigma B^{(3)})T_{24}=(\rho A^{(3)}+\sigma B^{(3)})T_{34}=0.
\end{aligned}$$

Hence

$$(\rho A^{(3)}+\sigma B^{(3)})T=T^*(3\,.\,L_0\dotplus 2\,.\,L_1\dotplus L_2),$$

where

$$T=[T_{14}, T_{24}, T_{34}, T_{13}, T_{23}, T_{12}]; \qquad T^*=[T^*_{13}, T^*_{23}, T^*_{12}]$$

Here T is a square matrix of non-zero determinant; its columns are the vectors (i, k, h) in a certain order. Similarly, we see that the columns of T^* are all the vectors $\{i, k, h\}$, which likewise make up a non-singular matrix. The canonical form of $\rho A^{(3)}+\sigma B^{(3)}$ is therefore

$$3\,.\,L_0\dotplus 2\,.\,L_1\dotplus L_2.$$

Instead of writing down the matrices T_{12}, T^*_{12}, . . . in full, it would have been sufficient to count the number of vectors (i, k, h) and $\{i, k, h\}$ which occur in all the chains together, and to convince ourselves that *all* vectors have been used. For, since no vector appears twice in the above chains, we can then evidently construct a non-singular square matrix which transforms the pencil into the aggregate of submatrices represented by the vector chains. The exact shape of the transforming matrix is irrelevant.

Now, consider the general case

$$\rho A+\sigma B\sim L_n,$$

i.e.

$$(\rho A+\sigma B)R=R^*L_n, \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

where

$$R = [r_0, r_1, \ldots, r_n] \quad \text{and} \quad R^* = [r_1^*, r_2^*, \ldots, r_n^*] \quad . \quad . \quad (21)$$

are non-singular matrices of degree $n+1$ and n respectively. Equation (20) is then equivalent to

$$Br_0 = 0;\ Ar_0 = Br_1 = r_1^*;\ Ar_1 = Br_2 = r_2^*;\ \ldots;\ Ar_{n-1} = Br_n = r_n^*,\ Ar_n = 0. \quad (22)$$

Next, we introduce $\binom{n+1}{p}$ column vectors of degree $\binom{n+1}{p}$:

$$(i_1, i_2, \ldots, i_p) = [r_{i_1}, r_{i_2}, \ldots, r_{i_p}]^{(p)}, \quad (0 \leqq i_1 < i_2 < \ldots < i_p \leqq n), \quad (23)$$

which are linearly independent because they are the columns of $R^{(p)}$ Similarly, we define $\binom{n}{p}$ vectors of degree $\binom{n}{p}$, namely,

$$\{j_1, j_2, \ldots, j_p\} = [r_{j_1}^*, r_{j_2}^*, \ldots, r_{j_p}^*]^{(p)}, \quad (1 \leqq j_1 < j_2 < \ldots < j_p \leqq n), \quad (24)$$

which are also linearly independent, being the columns of $(R^*)^{(p)}$. By (22) we have

$$A[r_{i_1-1}, r_{i_2-1}, \ldots, r_{i_p-1}] = B[r_{i_1}, r_{i_2}, \ldots, r_{i_p}] = [r_{i_1}^*, r_{i_2}^*, \ldots, r_{i_p}^*],$$

and, taking the pth compound of the matrices on either side, we get

$$A^{(p)}(i_1 - 1, i_2 - 1, \ldots, i_p - 1) = B^{(p)}(i_1, i_2, \ldots, i_p) = \{i_1, i_2, \ldots, i_p\} \quad (25)$$

(for $1 \leqq i_1 < i_2 < \ldots < i_p \leqq n$). The same method yields

$$B^{(p)}(0, i_2, \ldots, i_p) = 0; \qquad A^{(p)}(i_1, i_2, \ldots, i_{p-1}, n) = 0. \quad . \quad (26)$$

Equations (25) and (26) furnish the means of establishing vector chains for the pencil $\rho A^{(p)} + \sigma B^{(p)}$, a typical one being

$$\left.\begin{aligned}
&B^{(p)}(0, i_2, \ldots, i_p) = 0,\\
&A^{(p)}(0, i_2, \ldots, i_p) = B^{(p)}(1, i_2+1, \ldots, i_p+1) = \{1, i_2+1, \ldots, i_p+1\},\\
&A^{(p)}(1, i_2+1, \ldots, i_p+1) = B^{(p)}(2, i_2+2, \ldots, i_p+2) = \{2, i_2+2, \ldots, i_p+2\},\\
&\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots\\
&A^{(p)}(n-i_p-1, i_2+n-i_p-1, \ldots, n-1) = B^{(p)}(n-i_p, i_2+n-i_p, \ldots, n)\\
&\qquad\qquad\qquad = \{n-i_p, n-i_p+i_2, \ldots, n\},\\
&A^{(p)}(n-i_p, n-i_p+i_2, \ldots, n) = 0,
\end{aligned}\right\} \quad (27)$$

where

$$1 \leqq i_2 < i_3 < \ldots < i_p \leqq n \quad . \quad . \quad . \quad . \quad (28)$$

Each of these chains is of *type L* (§ 2), the initial and the final link being zero. The number of chains is equal to $\binom{n}{p-1}$, since the $p-1$ indices $i_2, i_3, \ldots, i_p$ are only subject to the condition (28). The chain (27) is of length $n - i_p$, i.e. $n - i_p + 1$ vectors of the kind (23), and $n - i_p$ vectors of the kind (24) occur in it (see § 2 (i, *b*)). It therefore gives rise to the submatrix

$$L_{n-i_p}$$

in the canonical form of the pencil. The smallest value for i_p is obviously $p-1$. Let

$$i_p = k.$$

According to (28) the remaining $p-2$ indices then have to satisfy the conditions

$$1 \leqq i_2 < i_3 < \ldots < i_{p-1} \leqq k-1.$$

Hence there are $\binom{k-1}{p-2}$ chains for which $i_p = k$, *i.e.* which are of length $n-k$; they correspond to the aggregate of submatrices

$$\sum_{k=p-1}^{n} \cdot \binom{k-1}{p-2} \cdot L_{n-k}. \qquad (29)$$

In order to show that (29) is already the canonical form of $\rho A^{(p)} + \sigma B^{(p)}$, we have only to prove that the number of rows and columns is the same in $\rho A^{(p)} + \sigma B^{(p)}$ as in (29), *i e.* equal to $\binom{n}{p}$ and $\binom{n+1}{p}$ respectively. Indeed, since L_k has k rows and $k+1$ columns, the pencil (29) has

$$\sum_{k=p-1}^{n} (n-k)\binom{k-1}{p-2} = \binom{n}{p} \qquad (30)$$

rows and

$$\sum_{k=p-1}^{n} (n-k+1)\binom{k-1}{p-2} = \binom{n+1}{p} \qquad (31)$$

columns The sums (30) and (31) can be evaluated by using the formulæ :[†]

$$\sum_{i=a}^{b} \binom{i}{s} = \binom{b+1}{s+1} - \binom{a}{s+1}, \qquad (32)$$

$$\sum_{i=a}^{b} i\binom{i-1}{s} = (s+1)\left\{\binom{b+1}{s+2} - \binom{a}{s+2}\right\} \qquad (33)$$

We have therefore proved the

THEOREM XIV:

$$(L_n;\ p) \sim \sum_{k=p-1}^{n} \cdot \binom{k-1}{p-2} \cdot L_{n-k}.$$

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[*] These formulæ may be obtained by comparing the coefficients of x^s in the identity

$$\sum_{i=a}^{b} (1+x)^i = \frac{1}{x}[(1+x)^{b+1} - (1+x)^a]$$

and in its derivative with respect to x.

Since $L'_n \sim M_n$, we obtain by transposition the

COROLLARY:

$$(M_n;\ p) \sim \sum_{k=p-1}^{n} \cdot \binom{k-1}{p-2} \cdot M_{n-k}.$$

This concludes the investigation of pencils based on compound matrices.

§ 6. PENCILS OF SCHLÄFLIAN MATRICES.

The treatment of Schläflian matrices does not lead to any new difficulties, since a close connection between Schlaflian and compound matrices will make it possible to refer to previous results. The procedure is exactly as in the last section (see Littlewood, 1935, § 4).

Let A be a matrix of type $m \times n$ and consider the transformation

$$y = Ax,$$

where y and x are column vectors of degree m and n respectively, viz.

$$y = \{y_1, y_2, \ldots, y_m\}; \qquad x = \{x_1, x_2, \ldots, x_n\}. \quad . \quad . \quad (1)$$

The $\binom{m+p-1}{p}$ products and powers of degree p which can be formed from the quantities $y_1, y_2, \ldots, y_m$ will then be aggregates of the $\binom{n+p-1}{p}$ products and powers of degree p constructed from the variables $x_1, x_2, \ldots, x_n$; we assume that these products and powers are arranged in dictionary order and write this transformation as

$$y^{[p]} = A^{[p]} x^{[p]}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Accordingly, $y^{[p]}$ and $x^{[p]}$ are column vectors and $A^{[p]}$ is a matrix of type $\binom{m+p-1}{p} \times \binom{n+p-1}{p}$; it is called the pth Schläflian matrix of A. We mention the following properties:—

$$(AB)^{[p]} = A^{[p]} B^{[p]}, \quad \text{(multiplicative law)} \quad . \quad (3)$$

$$[A']^{[p]} = (A^{[p]})'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Further, if A is square and of non-singular determinant, then so is $A^{[p]}$; this can easily be deduced from (3) (see p. 77).

Next, we consider the Schläflian matrix of a direct sum. Let

$$m = m_1 + m_2,$$
$$n = n_1 + n_2,$$

and

$$A = B \dotplus C = \begin{bmatrix} B & \cdot \\ \cdot & C \end{bmatrix},$$

where B is of type $m_1 \times n_1$ and C of type $m_2 \times n_2$. We shall then prove the

THEOREM XV:

Two permutation matrices P *and* R *can be found such that*

$$P(B \dotplus C)^{[p]}Q = \sum_{s=0}^{p} . B^{[p-s]} \times C^{[s]}.$$

Here P *and* Q *depend only on the number of rows and columns of* B *and* C *and not on their elements.*

Proof.—Let

$$u = Bx, \tag{5}$$

$$v = Cy, \tag{6}$$

where

$$u = \{u_1, u_2, \ldots, u_{m_1}\}, \qquad x = \{x_1, x_2, \ldots, x_{n_1}\},$$
$$v = \{v_1, v_2, \ldots, v_{m_2}\}, \qquad y = \{y_1, y_2, \ldots, y_{n_2}\}$$

are column vectors of degrees m_1, n_1, m_2, n_2 respectively, and the elements of x and y are independent variables. Put

$$\begin{bmatrix} x \\ y \end{bmatrix} = \{x, y\}; \qquad \begin{bmatrix} u \\ v \end{bmatrix} = \{u, v\}.$$

We have then

$$\{u, v\} = A\{x, y\},$$

and, taking the pth Schläflian of either side,

$$\{u, v\}^{[p]} = A^{[p]}\{x, y\}^{[p]}. \tag{7}$$

Here $\{x, y\}^{[p]}$ is a column vector whose elements are the $\binom{n}{p}$ products and powers which can be formed from the elements of x and y. Obviously, the vectors

$$\{z\} = \{x^{[p]}, \quad x^{[p-1]} \times y, \quad x^{[p-2]} \times y^{[2]}, \quad \ldots, \quad y^{[p]}\} \tag{8}$$

and $\{x, y\}^{[p]}$ have the same elements apart from the order. We can therefore find a permutation matrix R such that

$$\{x, y\}^{[p]} = R\{z\}. \tag{9}$$

Similarly,

$$P\{u, v\}^{[p]} = \{w\}, \tag{10}$$

where

$$\{w\} = \{u^{[p]}, \quad u^{[p-1]} \times v, \quad u^{[p-2]} \times v^{[2]}, \quad \ldots, \quad v^{[p]}\}. \tag{11}$$

Hence, by (7) and (9) and (10),

$$\{w\} = PA^{[p]}R\{z\}.$$

On the other hand, we evidently have

$$\{w\}=\left(\sum_{s=0}^{p}.\,B^{[p-s]}\times C^{[s]}\right)\{z\}$$

by (5) and (6). Since there is no linear relation between the elements of $\{z\}$, it follows that

$$PA^{[p]}R=\sum_{s=0}^{p}.\,B^{[p-s]}\times C^{[s]}.$$

If

$$H=\rho A+\sigma B$$

is a given pencil, we define a new pencil

$$[H;\ p]=\rho A^{[p]}+\sigma B^{[p]}.$$

As before, we can show that

$$[H^*;\ p]\sim[H;\ p]$$

if

$$H^*\sim H.$$

We may therefore assume that H appears in canonical shape. Again, if

$$H=H_1\dotplus H_2=\rho(A_1\dotplus A_2)+\sigma(B_1\dotplus B_2),$$

we have by the last theorem:

$$P(A_1\dotplus A_2)^{[p]}R=\sum_{s=0}^{p}.\,A_1^{[p-s]}\times A_2^{[s]}, \quad . \quad . \quad . \quad (12)$$

$$P(B_1\dotplus B_2)^{[p]}R=\sum_{s=0}^{p}.\,B_1^{[p-s]}\times B_2^{[s]}, \quad . \quad . \quad (13)$$

with the same matrices P and R in (12) and (13). Hence, multiplying (12) by ρ and (13) by σ and adding, we get

$$[H_1\dotplus H_2;\ p]\sim\sum_{s=0}^{p}([H_1;\ p-s]\mid[H_2;\ s]).$$

It is therefore sufficient to consider the pencil

$$[H;\ p]$$

when H is one of the elementary submatrices L, M, N, Q, $W(\alpha)$. As regards the last two of these cases, Littlewood (*loc. cit.*, § 4) has proved the following theorems:—

THEOREM XVI:

$$[W_n(\alpha);\ p]\sim(\underset{n+p-1}{W}(\alpha);\ p)\sim\sum_{s=0}^{\frac{t-1}{2}}.\,c_s\,.\,\underset{r-s}{W}(\alpha^p),$$

where t=p(n − 1) + 1, *and* c_s *is the number of partitions of* s *minus the number of partitions of* s − 1 *into* ≦ p *parts each* ≦ n − 1.

THEOREM XVII:

$$[Q_n;\ p] \sim (Q_{n+p-1};\ p) \sim \sum_{s=0}^{n-1} \cdot \binom{p-2+s}{p-2} \cdot Q_{n-s}.$$

By transposing and interchanging ρ and σ we obtain the

COROLLARY:

$$[N_n;\ p] \sim (N_{n+p-1},\ p) \sim \sum_{s=0}^{n-1} \cdot \binom{p-2+s}{p-2} \cdot N_{n-s}.$$

We shall now show that similar relations hold for the singular submatrices.

THEOREM XVIII:

$$[L_n;\ p] = (L_{n+p-1};\ p) \sim \sum_{s=0}^{n} \cdot \binom{s+p-2}{p-2} \cdot L_{n-s};$$

and by transposition,

COROLLARY:

$$[M_n;\ p] = (M_{n+p-1};\ p) \sim \sum_{s=0}^{n} \cdot \binom{s+p-2}{p-1} \cdot M_{n-s}.$$

Proof.—By § 1 (2) we have

$$L_n = \rho F_n + \sigma G_n,$$

where

$$F_n = [\ .\ I_n]_{n,\ n+1}; \qquad G_n = [I_n\ .\ ]_{n,\ n+1}. \qquad (14)$$

Hence

$$[L_n;\ p] = \rho F_n^{[p]} + \sigma G_n^{[p]},$$

$$(L_{n+p-1};\ p) = \rho F^{(p)}_{n+p-1} + \sigma G^{(p)}_{n+p-1}.$$

In order to prove Theorem XVIII we have to show that

$$F_n^{[p]} = F^{(p)}_{n+p-1}, \qquad (15)$$

$$G_n^{[p]} = G^{(p)}_{n+p-1}. \qquad (16)$$

Consider the transformations:

$$(a)\ \ y = F_n x = [I_n\ .\ ]x; \qquad (b)\ \ y = G_n x = [\ .\ I_n]x, \qquad (17)$$

where

$$x = \{x_1, x_2, \ldots, x_{n+1}\}; \qquad y = \{y_1, y_2, \ldots, y_n\}. \qquad (18)$$

We write (17) down in full:

$$\left.\begin{array}{llll} (a)\ \ F_n: & \begin{array}{l} y_1 = x_1 \\ y_2 = x_2 \\ \ \ \ldots \\ y_n = x_n \end{array} & (b)\ \ G_n: & \begin{array}{l} y_1 = x_2 \\ y_2 = x_3 \\ \ \ \ldots \\ y_n = x_{n+1} \end{array} \end{array}\right\} \qquad (19)$$

Now, $F_n^{[p]}$ is the matrix which expresses the products

$$y_1^{j_1} y_2^{j_2} \ldots y_n^{j_n} \quad (j_1 + j_2 + \ldots + j_n = p;\ j_v \geqq 0) \tag{20}$$

in terms of the products

$$x_1^{i_1} x_2^{i_2} \ldots x_{n+1}^{i_{n+1}} \quad (i_1 + i_2 + \ldots + i_{n+1} = p;\ i_v \geqq 0). \tag{21}$$

We associate the products (20) with the partitions

$$(1^{j_1}, 2^{j_2}, \ldots, n^{j_n}), \quad . \quad . \quad . \quad . \tag{22}$$

the p parts of which are arranged in non-decreasing order. If we add zero to the first part, unity to the second part, two to the third part, . . . , $p-1$ to the last part, we obtain a partition

$$(h_1, h_2, \ldots, h_p) \quad (1 \leqq h_1 < h_2 < \ldots < h_p \leqq n+p-1). \tag{22a}$$

These partitions are in 1-1 correspondence to (22) and hence also to (20). We may therefore put

$$y_1^{j_1} y_2^{j_2} \ldots y_n^{j_n} = (h_1, h_2, \ldots, h_p)_y. \quad . \quad . \quad . \tag{23}$$

In the same way we introduce the notation:

$$x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} x_{n+1}^{i_{n+1}} = (k_1, k_2, \ldots, k_p)_x, \quad . \quad . \tag{24}$$

where the partition $(k_1, k_2, \ldots, k_p)$ is derived from

$$(1^{i_1}, 2^{i_2}, \ldots, (n+1)^{i_{n+1}})$$

exactly as (22*a*) is obtained from (22). The k's satisfy the inequalities

$$1 \leqq k_1 < k_2 < \ldots < k_p \leqq n+p.$$

According to (19) (*a*) the effect of $F_n^{[p]}$ can now be described by the equations

$$y_1^{j_1} y_2^{j_2} \ldots y_n^{j_n} = x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$$

or

$$F_n^{[p]}:\ (h_1, h_2, \ldots, h_p)_y = (h_1, h_2, \ldots, h_p)_x. \quad . \quad . \tag{25a}$$

By (19) (*b*) we get in the same manner for $G_n^{[p]}$:

$$y_1^{j_1} y_2^{j_2} \ldots y_n^{j_n} = x_2^{j_1} x_3^{j_2} \ldots x_{n+1}^{j_n}$$

or

$$G_n^{[p]}:\ (h_1, h_2, \ldots, h_p)_y = (h_1+1, h_2+1, \ldots, h_p+1)_x. \quad . \quad . \tag{25b}$$

For the discussion of the compound matrices $F_{n+p-1}^{(p)}$ and $G_{n+p-1}^{(p)}$ we consider the transformations:

$$(a)\ y_{(r)} = F_{n+p-1} x_{(r)}; \qquad (b)\ y_{(r)} = G_{n+p-1} x_{(r)}, \quad (r = 1, 2, \ldots, p), \tag{26}$$

where $x_{(r)}$ and $y_{(r)}$ are column vectors of degrees $n+p$ and $n+p-1$.

Let

$$x_{(r)} = \{x_{1r}, x_{2r}, \ldots, x_{n+p,\, r}\} \quad \text{and} \quad y_{(r)} = \{y_{1r}, y_{2r}, \ldots, y_{n+p-1,\, r}\}.$$

As is well known, $F^{(p)}_{n+p-1}$ and $G^{(p)}_{n+p-1}$ are the matrices which express the determinants

$$[h_1, h_2, \ldots, h_p]_y = \begin{vmatrix} y_{h_1 1} & \cdot & y_{h_1 p} \\ \cdot & \cdot & \cdot \\ y_{h_p 1} & \cdot\ \cdot & y_{h_p p} \end{vmatrix} \qquad (1 \leqq h_1 < h_2 < \ldots < h_p \leqq n+p-1)$$

in terms of the determinants

$$[k_1, k_2, \ldots, k_p]_x = \begin{vmatrix} x_{k_1 1} & \cdot\ \cdot\ \cdot & x_{k_1 p} \\ & \cdot & \cdot\ \cdot \\ x_{k_p 1} & \cdot\ \cdot & x_{k_p p} \end{vmatrix} \qquad (1 \leqq k_1 < k_2 < \ldots < k_p \leqq n+p)\,.$$

when the y's are transformed into the x's by means of the matrices F_{n+p-1} and G_{n+p-1}, *i.e.* when

$$(a)\ \ y_{hr} = x_{hr}\ \ (\text{for } F_{n+p-1})\,; \qquad (b)\ \ y_{nr} = x_{h+1,\, r}\ \ (\text{for } G_{n+p-1}).$$

The transformation of the determinants may therefore be described as follows:—

$$F^{(p)}_{n+p-1}:\ [h_1, h_2, \ldots, h_p]_y = [h_1, h_2, \ldots, h_p]_x, \qquad \text{(27a)}$$

$$G^{(p)}_{n+p-1}:\ [h_1, h_2, \ldots, h_p]_y = [h_1+1, h_2+1, \ldots, h_p+1]_x. \qquad \text{(27b)}$$

A comparison between (25) and (27) shows that the transformations corresponding to $F^{[p]}_n$ and $G^{[p]}_n$ differ only in the notation of the variables from those associated with $F^{(p)}_{n+p-1}$ and $G^{(p)}_{n+p-1}$ The respective matrices are therefore identical and Theorem XVIII is proved.

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(*Issued separately July* 11, 1936.)

VII.—**Fossil Pollen in Scottish Tertiary Coals.** By **J. B. Simpson,** B.Sc.Agri., D.Sc. (With Three Plates.)

(MS. received March 17, 1936. Read June 1, 1936)

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INTRODUCTION.

ASSOCIATED with the volcanic rocks of Tertiary age that bulk so largely in the Mull, Morvern, and Ardnamurchan districts of Argyllshire, western Scotland, are occasional sedimentary deposits of insignificant proportions. The sediments, for the most part, consist of sandstones and conglomerates, but there may also be shaly bands and thin, impersistent seams of lignites (*Mull Memoir*, 1924, Chap. III). These lignites have been found to possess an abundant pollen-content of great interest and considerable variety, and this paper is a preliminary account of some of the forms isolated. It has not yet been possible to make a detailed examination of each seam throughout its entire thickness and thus get a complete picture of its pollen-aspect—but it is hoped to attempt this interesting task in the near future.

GEOLOGICAL HORIZON OF THE LIGNITES.

One of the lignite horizons almost immediately precedes the Basal Mudstone (*Mull Memoir*, 1924, p. 59) of the volcanic series, a thin but widely distributed ash which heralded the onset of Tertiary vulcanism in this area. Visible below the mudstone at several localities are a few (4–20) ft. of strata, mainly sandstone, with lignite in the form of streaks or wisps in the sandstone or as thin seams associated with underclays. The lignite-bearing beds may rest upon rocks of widely varying ages; but they are certainly younger than the Senonian division of the Cretaceous —for at Ben Iadain, Morvern, they rest upon strata of this age (Pringle and Manson, 1934, p. 83). The Ben Iadain section (Manson and Simpson, 1923, p. 96), which is probably the best exposed, is detailed below:

	Ft	In
Basal mudstone	1	6
Sandstone with streaks of lignite	2	0
Sandstone (round grains) reddish brown passing down to white; occasional plant fragments	1	9
Lignite streak	..	
Hard white fireclay, full of roots .	0	4
Lignite with rounded sand-grains . .	0	2
Silicified chalk with flints (SENONIAN)	2	6

It will be noted that the lignite is associated with an underclay (fireclay), a fact that points to its accumulation *in situ* These post-Senonian beds have yielded no satisfactory palæontological evidence to show that they are of Tertiary rather than Cretaceous age. The chalk rock on which they rest, however, is highly silicified and fissured, the result, as Bailey (*Mull Memoir*, 1924, p 54) points out, of uplift and subærial weathering before the accumulation of the lignite beds. It is reasonable, therefore, to accept this break as marking the end of the Cretaceous, though the time-gap represented need not be great.

The other sedimentary horizons occur as intercalations in the lava-flows at various localities in Mull—Ardtun, Carsaig, etc. (*Mull Memoir*, 1924, p. 61). The outcrops at these localities are isolated from one another, but they probably are on or about the same horizon, underlain by a number of lava-flows whose combined thickness is difficult to estimate accurately, but which is not likely to exceed 500 ft. It is certain that the sediments are all very much nearer the base than the top of the volcanic succession, which amounts to several thousands of feet of lava.

In thickness the intercalations range from a few inches in some instances to 60 ft. at Ardtun (*Mull Memoir*, 1924, p. 62). The Ardtun exposure is well known on account of its leaf-beds whose finely preserved contents have been described by several workers, Forbes (1851), Gardner and Ettinghausen (1882), Gardner (1886, 1887) and, most recently, Seward and Holttum (*Mull Memoir*, 1924, Chap. IV), to whose account readers are referred for a complete bibliography relating to Ardtun.

The lignites discussed here were obtained from three localities:

(1) Ardslignish, Ardnamurchan, one-inch Map 52, Lat. 56° 41′ 30″, Long 5° 57′ 35″ W.

(2) Shiaba, Bunessan, Mull, one-inch Geological Map 43, Lat. 56° 17′ 50″, Long. 6° 7′ 50″ W.

(3) Bremanoir, Bunessan, Mull, one-inch Geological Map 43, Lat. 56° 20′ 40″, Long. 6° 13′ 0″ W.

The specimen from Ardslignish is from the infrabasaltic horizon. The lignite, 1–2 ft. thick, occurs about 3 ft. below the Basal Mudstone, and rests upon 2–4 in. of underclay overlying pre-Cambrian strata. Shiaba and Bremanoir are localities within the lava outcrops 4 miles apart, and the sediments are intercalated between lava-flows.

PREPARATION OF MATERIAL.

The method practised in extracting the pollen is essentially that described by Raistrick (1934, pp. 142–153) in his studies on the microspore-content of Carboniferous coals. In the case of the lignites, however, it is not necessary to powder the coal beyond the stage of coarse granules (say 2·5 mm. particles); nor is it necessary to continue the treatment with the Schulze's Solution and the alkali for periods longer than 18 hours in each case.

For microscopic examination the pollen-bearing residue so obtained was mounted in glycerine-jelly stained with safranin. It was found in every instance to be extremely rich in pollen in an excellent state of preservation. Spores of non-flowering plants are also present, but in no great abundance. These are not discussed in the present paper.

DESCRIPTION OF SPECIMENS.

The similarity of the fossil pollen-forms to those of modern living species is usually so marked that the appropriate modern generic name has in general been used, but preceded by "cf." in cases where doubt exists as to complete identity with the modern genus to which the grain appears to be, however, obviously related.

At this early stage in these studies it was felt inadvisable to coin new generic names which may at a later date prove unnecessary and have to be discarded. No attempt has been made to name the forms specifically as studies of modern pollen provide abundant proof that in general the value of pollen as a specific determinant is low.

GYMNOSPERMÆ.

GINKGOALES.

Ginkgo sp. (Pl. I, figs. 1, 2).—Grains elliptical or nearly spherical in outline, about 30 μ in length, breadth variable, from about half the length upwards; exine slightly roughened, but this feature not always notable; longitudinal furrow distinct, with open rounded ends, wavy margin of furrow and body-wall generally obvious.

These grains very closely resemble those of the modern *Ginkgo biloba*. The wavy margin of furrow and body-wall is obvious in Pl. I, fig. 1, while fig 2 of this plate illustrates the broadly elliptical form that the grains sometimes take. They are not infrequent in the Shiaba and Bremanoir seams, but have not been noted at Ardslignish. *Ginkgo adiantoides* has been recorded at Ardtun by Gardner (1886, p 99, pl. xxv), and Seward and Holttum (*Mull Memoir*, 1924, p. 74).

CONIFERALES.

Abietineæ.

Pollen grains provided with two conspicuous air-sacs or bladders (winged pollen grains) as in certain genera of this division of the conifers are common in the interbasaltic lignites, but have not been noted in the infrabasaltic seam. The generic separation of the various forms is sometimes difficult unless the grains are favourably presented for examination.

The following genera are certainly present:—

Pinus sp. (Pl. I, figs. 3, 4, 5; Pl. II, fig. 1).—Body breadth (distance between dorsal roots of bladders) 65–70 μ; cap (dorsal surface) almost circular in outline, reticulate-granular, bounded by a well-marked marginal ridge; ridge forming a distinct projection at dorsal roots of bladders, additional wrinkling of the exine of cap at this point evident; bladders relatively large, almost 50 μ broad, not greatly, if at all, flattened; reticulation of bladder wall distinct, meshwork open and regularly polygonal towards bladder tips.

Pl. I, fig. 3, is an end view of a grain, and shows clearly the ventral situation of the bladders (compare with *Cedrus*, Pl. II, fig. 2). The well-developed marginal ridge, and projection at dorsal root of bladder are also evident. Pl. I, fig. 4, illustrates the wrinkling of the dorsal surface towards the bladder roots. Pl. I, fig. 5, and Pl. II, fig. 1, are views of the dorsal and ventral aspects of grains showing the outline of the cap and size and proportions of the bladders.

In their detailed characteristics these grains resemble the pollen of a number of the modern species of *Pinus* The body size of the modern species varies between 50 and 70 μ, so that this fossil species approaches the upper limit of this range; but even in this respect it can be matched by quite a number of the modern forms, e.g. *pinea*, *pinaster*, and *halepensis*.

Seward and Holttum (*Mull Memoir*, 1924, p. 74) record from the

Carsaig deposit a species of five-needled pine which they compare with *P strobus.* It cannot be said that the fossil pollen here described shows a marked resemblance to that of this species.

Pinus pollen has been recorded only from the interbasaltic seams

Cedrus sp. (Pl. II, figs. 2, 3).—Body circular or transversely elongate, 52–65 μ, exine of cap finely reticulate or granular, marginal ridge not evident; bladders distinctly laterally placed, flaccid, often collapsed and partially enveloping ventral surface of grain, reticulation of bladders merging with that of the body, meshwork crumpled, and pattern not distinctly polygonal.

Cedrus pollen is the dominant coniferous type in the Shiaba coal. Pl. II, fig. 2, illustrates the grain in a semi-expanded condition. The outline of back and bladders forms an almost unbroken curve. The relatively smaller size of the bladders and their more lateral position are striking compared with the corresponding features in *Pinus* (Pl. I, fig. 3). In Pl II, fig. 3, the *Cedrus* pollen is viewed from the ventral side in the highly characteristic contracted condition. In this instance the bladders are invaginated and have moulded themselves on the ventral surface. The outline of the grain becomes almost circular, and the margin of the dorsal surface is visible through the bladder membranes overlapped by about half the bladder width. In *Pinus* (Pl. II, fig. 1) the margin of the cap is at the outside edge of the bladders. The meshwork of the bladders also appears crumpled, blurred, and less regularly polygonal in comparison with that of *Pinus.* These grains compare very closely with the pollen of the modern cedars.

Fossil records of *Cedrus* in early Tertiary rocks are not numerous. Fossil cedar-wood (*Cedroxylon*) has been recorded by Walton (1927, p. 245) and by Høeg (1931, p. 377) from Tertiary strata in Greenland. Wodehouse (1933, p. 490) has recorded under the name *Cedripites eocenicus,* what he regards as cedar-pollen in the Tertiary shales of the Green River formation. Recently Seward and Conway (1935, p. 10) have described a cedar-like seed *Pityospermum cedroides* from strata probably of Eocene age from Kingigtok, Greenland.

Abies sp. (Pl. II, fig. 4).—Grains large, body breadth exclusive of bladders 96 μ, breadth including bladders 125 μ; exine of dorsal surface thick but not markedly rough in texture, appears to merge gradually with the ventral surface, no apparent marginal ridge; bladders set at an angle to cap, moderately large.

Grains of this type are not numerous, but catch the eye by reason of their large size. They have been recorded only from the Shiaba seam. Their resemblance is most close to *Abies* of the modern genera. The bladders are probably larger than is usual in this genus, but in the fossil grains they appear to be slightly invaginated or compressed dorso-ventrally, and it is difficult to make an exact measurement of their dimensions.

Fossil records of *Abies* are rare. Wodehouse (1933, p 489) records typical pollen of this type in the Green River beds.

Podocarpineæ.

Grains corresponding in character to those of *Podocarpus* are not common, but two, finely preserved, have come to light in the Shiaba seam.

Podocarpus sp. (Pl. II, fig. 5).—Body oval, narrowed transversely, 35 μ by 45 μ; exine of cap smooth, hyaline, with crinkled border; bladders large, almost touching, slightly unequal in size, equalling or exceeding body width and markedly exceeding body length; junction with body on ventral side extending whole length of grain, sharply defined; reticulation of bladder walls pronounced, but details of pattern not well preserved.

This type is distinguished from the winged grains of the Abietineæ by the smaller body size, the relatively enormous bladders, and the smooth texture of the exine of the body-wall. It resembles modern *Podocarpus* pollen in the relative proportion of bladders and body, the smooth exine of the dorsal surface, and the conspicuous crinkled margin of the cap. It differs from all modern forms in its larger size, its dimensions all round being from one-third to one-half as large again. Three species of *Podocarpus* have been recorded from the Mull beds by Gardner (1886, p. 97; 1887, p. 289), and one of these also by Seward and Holttum (*Mull Memoir*, 1924, p. 76).

Taxodiinæ, *Cupressineæ*, *Taxeæ.*

Spherical, smooth-walled or slightly roughened grains, 25–35 μ in diameter, often almost completely split into two parts, are not uncommon in the preparations of the Shiaba and Bremanoir seams. They bear a distinct resemblance to the pollen of members of the coniferous classes indicated in the heading, but it has not been found possible to separate them generically with any degree of certainty.

Their record, however, may be of value in similar studies elsewhere, and an example is accordingly figured (Pl. III, fig. 19). It is a split grain 36 μ long, with possibly a slightly roughened exine, and with the margins of the split portions reflexed. It might be compared with the pollen of *Glytostrobus* sp. (Wodehouse, 1933, p. 494), but its more precise identification is for the present left an open question.

ANGIOSPERMÆ.

MONOCOTYLEDONES.

Liliaceæ.

Smilax? sp. (Pl. III, fig. 6).—Grains broadly oval or spherical, 27–35 μ in diameter; wall of grain thin, carrying numerous short spinose projections which are somewhat patchy in their distribution.

Grains with these characters have been noted in the Ardslignish and Shiaba seams. They most closely resemble the pollen of the monocotyledonous genus *Smilax*, e g. *S China*. Very similar grains have been recorded by Wodehouse (1933, p. 500) in the Green River shales under the generic name of *Smilacipites*. He points out the resemblance between the pollen of *Smilax* and *Peltandra*, a genus of the Araceæ. Attention might also be drawn to the somewhat similar pollen of *Sassafras* sp While recognizing the uncertainty that these resemblances introduce, the probable relationship of the fossil grains to *Smilax* has been emphasized, since actual comparison with the various forms most strongly suggest this conclusion.

DICOTYLEDONES.

Juglandaceæ.

Engelhardtia sp. (Pl. III, fig. 18).—Grains flattened, spherical-triangular in outline, 18–19 μ; pores 3, at corners, slightly bulging, each pore surrounded by a circular clear space of considerable area; remainder of surface finely but distinctly granular.

These grains closely resemble those of *Engelhardtia spicata* In size and shape they also approach *Betula* pollen, but the latter is larger, the pores bulge much more markedly, the clear space round the pore is much smaller, and there is not such a distinctly granular central area. This species is fairly common in the Ardslignish seam, but has not been noted elsewhere.

Betulaceæ.

Alnus spp. (Pl. II, figs. 8, 9).—Grains regularly polygonal in outline, 17–30 μ across; pores 4–5 in number, conspicuous, slightly protruding, elliptical or slit-like, linked by conspicuous curved bands; surface of grain smooth.

Pollen of *Alnus* type occurs both in the infra- and interbasaltic seams. It is, however, decidedly more common in the Shiaba seam than in that at Ardslignish. In general characters the fossil alders closely resemble their modern successors. They are polygonal, generally 4–5-sided, and exhibit very clearly the thickened curving bands between the pores. They differ from the modern forms in possessing a much greater range in size. The 4- and 5-sided grains vary between 17 μ and 30 μ, generally between 25 μ and 30 μ, and are hardly to be distinguished from modern specimens.

Cf *Alnus* sp. (Pl. II, figs. 6, 7)—Two very similar specimens, one from Ardslignish (Pl. II, fig. 6), the other from Shiaba (Pl. II, fig. 7), are distinguished by measuring practically 50 μ across. In one 6 pores and in the other 7 pores are clearly visible, while in both there is a suggestion of another pore within a centrally situated thickened ring.

According to Wodehouse (1935, p. 309) the size-range of modern alder pollen is 19 to 27 μ, while the maximum number of pores is 7. In respect of size, therefore, and also possibly in respect of number of pores, these fossil forms differ markedly from the modern types and the other fossil *Alnus* described above. It is even within the bounds of possibility that they are merely abnormal "giant" grains with supernumerary pores.

Forbes (1851, p. 103) records a doubtful *Alnites* from Ardtun. According to Seward and Holttum (*Mull Memoir*, 1924, p. 78), however, the specimen is not complete, and upon this and similar material they refrain from making "any definite suggestion as to the true nature of this species." Fossil alder pollen has been recorded in American Tertiary rocks by Wodehouse (1933, p. 508), and it is often very abundant in the Tertiary brown coals of Germany (Potonié, 1931, p. 331).

Corylus? sp. (Pl. III, fig. 20)—Grains triangular to spherical in outline, 25–35 μ in diameter; pores 3, not protruding, circular, body-wall apparently not thickening towards edge of pore; exine smooth or very faintly granular.

Grains of this nature are not infrequent in the samples from all the horizons examined. Unfortunately these characters are shared in some degree by the pollen of other genera, e g. *Myrica*, *Ostrya*, *Carpinus* (*viminea*), any of which might be expected here. These grains have been tentatively assigned to *Corylus*, because this genus (*Corylites*) is the only one of those mentioned that has been recorded from Mull (Seward and Holttum, *Mull Memoir*, 1924, p. 79). The figured grain is from the Shiaba seam, and is typical of its kind.

Ulmaceæ.

Planera sp. (Pl. II, figs. 10, 11).—Grains 4- or 5-sided or rounded in outline, 25–35 μ; exine smooth on outer surface, but a reticulate pattern developed in, or on inner surface of, wall; pores 4 or 5, conspicuous, often flecked, elliptical, convergent in pairs; curved linear thickenings linking pores usually conspicuous at least along part of their length.

Grains corresponding to this description are numerous in the Shiaba and Bremanoir seams, but have not been noted in the infrabasaltic seam from Ardslignish. In size they are mostly all about 30 μ, and vary only within narrow limits The majority of the grains are 4-pored. The pores are fairly large, elliptical in outline (Pl II, fig 11), with the major axis about twice the minor. The major axes of the pores converge in pairs towards the opposite sides of the grain. The black fleck in the centre of the pore is often perfectly distinct. In the pollen preparations the 4-pored grains generally lie so that the pores are on the periphery (Pl II, fig. 10), and so placed that the indentations of one of the convergent pairs are deeper than those of the other. The resulting squarish outline, broken at the corners by two deep and two shallow notches, is highly characteristic of these grains. Curved linear thickenings connecting the pores are also generally conspicuous though not so pronounced or so regularly developed as in *Alnus*. Often, as in Pl II, fig 11, they appear as divergent streaks from the neighbourhood of a pore.

The detailed resemblance of these grains to the pollen of the modern *Planera aquatica* is extremely close They are distinguished from those of the allied genera *Ulmus* and *Zelkowa* by possessing generally 4, not 5, pores, and by the presence of the thickened bands between the pores It should be pointed out that a suggestion of the latter is sometimes faintly discernible in *Ulmus montana*, and, according to Wodehouse (1935, p. 386), in *U. crassifolia.*

Cf. *Pteroceltis* sp. (Pl. III, fig. 5)—Grains spherical-triangular in outline, 27 μ by 31 μ; pores 3 at corners, circular, not projecting above general outline of grain, small fleck preserved in centre of pore; exine granular, granulation somewhat unevenly distributed but up to margin of pore, with suggestion of circular thickening.

This grain bears a very close resemblance to the pollen of the related genera *Celtis* and *Pteroceltis*, particularly to that of *Pteroceltis Tartarinowii*. The pollen of these genera, however, cannot always be satisfactorily separated (Wodehouse, 1935, p. 390), and in this instance cf. *Pteroceltis* must be considered as including both *Celtis* and *Pteroceltis*. In the figure (Pl. III, fig 5) the non-projecting pores, the marked but uneven granulation, and the fleck in the aperture of one of the pores, are all clearly seen. It differs from the modern forms with which it has been compared in being slightly more triangular in outline

The illustrated grain has been observed in the Ardslignish seam. The form is doubtfully present in the Shiaba seam

Magnoliaceæ.

Magnolia spp. (Pl. I, figs. 6, 7, 8, 9).—Grains oval in outline, 40–70 μ length, breadth variable, usually half to two-thirds of the length; exine smooth or granular-reticulate, conspicuous furrow extending full length of grain or almost so, furrow ends pointed, margins distinct, **V**-shaped rarely **U**-shaped in depth, furrow membrane sometimes faintly granular

Grains with these characters are present in the samples from all horizons. They occur in the Ardslignish (infrabasaltic) seam in great numbers, but only occasionally in the Shiaba and Bremanoir coals. The Ardslignish examples display considerable variation in size, and it is certain that two species or two groups of species are represented—with pollen grains averaging 50 μ and 65 μ in length respectively.

The specimens from Shiaba and Bremanoir (interbasaltic) do not show a greater length variation (42–55 μ) than may be encountered in a modern species of *Magnolia*. In the Shiaba specimens (Pl. I, fig. 8) the exine is smooth or faintly granular, whereas the specimens from the older (Ardslignish) seam (Pl. I, figs. 6, 7, 9) are generally conspicuously granular. However, even in the latter respect, they are matched by some of the modern species, e.g *Campbelli* and *Kobus*.

In the forthcoming Summary of Progress of the Geological Survey for the present (1936) year, Professor T. Johnson reports the presence of *Magnolia* leaves in a recently discovered interbasaltic deposit in the Island of Skye of approximately the same age as the Mull beds.

Hamamelidaceæ.

The pollen characters of the members of this order show considerable variation, and, for the purposes of pollen study, the genera may conveniently be assembled in the following groups:—

(*a*) Oblately spherical 3-furrowed grains, 15–35 μ in equatorial diameter, with markedly reticulated exine. Furrows deeply **V**-cleft, extending almost to the poles, each furrow occupied by a pore which may be very large; pore boundary often ill-defined, pore membrane generally conspicuously flecked with stumpy black rod-shaped bodies —*Bucklandia*, *Corylopsis*, *Disanthus*, *Eustigma*, *Fortunearia*, *Fothergilla*, *Hamamelis*, *Loropetalum*, *Rhodoleia*, *Sinowilsonia*, *Tetrathyrium*, and *Trichocladus*

(*b*) Oblately spherical 3-furrowed grains, 35 μ in diameter, exine very coarsely reticulate, furrows much shorter than in (*a*), usually about a quarter of circumference.—*Dicoryphe macrophylla*.

(*c*) Highly oblate, almost flattened grains, triangular in outline, 40 μ across, reticulate, pores 3, large, conspicuously flecked —*Parrotia*.

(*d*) Pentagonal or roughly spherical, 35 μ in diameter, with 5 very large flecked pores.—*Sycopsis*.

(*e*) Spherical, multi-pored, reticulate or pitted grains, 35 μ in diameter. —*Altingia*, *Liquidambar*.

(*f*) Spherical, finely reticulate or pitted grains, variable in size, 20–40 μ in diameter, flaccid, collapsing or contracting in a regular manner that suggests an obscurely furrowed grain —*Distylium racemosum*.

(*g*) Grains in tetrahedral tetrads, diameter of tetrad 35 μ, grains poreless, furrowless, exine finely reticulate —*Myrothamnus*.

The genera in group (*a*) can be further distinguished by differences in size, and particularly by differences in the pattern of the reticulated exine

Hamamelid pollen is extremely abundant in the Ardslignish seam, and not infrequent in the Shiaba and Bremanoir samples. As a rule, the furrow-membrane is not preserved or not clearly visible in the fossil condition, and the generic identification depends upon the reticulated portion of the exine. The genera identified belong to groups (*a*), (*b*), and (*f*). No examples of (*c*), (*d*), (*e*), and (*g*) have been noted so far.

Bucklandia sp. (Pl. III, figs. 3, 7).—Spherical, deeply 3-cleft, markedly oblate, coarsely reticulate, 26–35 μ in diameter; meshes somewhat irregular in shape, elongated meridionally in equatorial region of grain, equidimensional and smaller at poles.

Except that it is distinctly larger the fossil form very closely resembles the modern *Bucklandia populnea* (Pl. III, fig. 4), which has been included for comparison. The typical oblate shape of the grain is well illustrated in Pl. III, fig 7. In this specimen the furrow membrane is preserved and bears a small aperture which has the appearance of a pore, but about which it is not possible to be certain. It may be an accidental rent in the tissue. The elongation of the mesh in the equatorial region is also apparent in this specimen. In its curiously irregular mesh pattern *Bucklandia* is distinctive amongst the hamamelid genera *Bucklandia* has been observed only in the Ardslignish (infrabasaltic) seam.

Corylopsis sp (Pl. III, figs. 9, 10)—Spherical, deeply 3-cleft, markedly oblate, 25–35 μ in equatorial diameter, finely reticulate, meshes equidimensional, slightly irregular but not decreasing in size towards poles

These grains have been observed in the Ardslignish and Shiaba seams. They compare very closely with those of the genus *Corylopsis*, e g. *C. sinensis* (Pl. III, fig. 8). They also resemble, though not so markedly, *Fothergilla*. The reticulation of the exine is much finer than in *Bucklandia*. Mathiesen (1932) records probable *Corylopsis* wood (*Corylopsites grœnlandicus*) from Tertiary beds in Greenland

Fortunearia sp. (Pl. III, fig. 11).—Spherical, 3-cleft, slightly oblate, 26 μ in diameter, very finely reticulate, mesh shape apparently equidimensional

This grain is not so markedly oblate as other types. The reticulation, also, is very much finer than in *Corylopsis*. In all its characters it shows a close resemblance to the pollen of *Fortunearia sinensis* (Pl. III, fig. 12) This type has been observed with certainty only in the Ardslignish seam, in which it is common.

Loropetalum sp. (Pl. III, fig. 1)—Spherical, 3-cleft, 17½ μ in diameter, finely reticulate, meshes equidimensional, mesh size not lessening towards poles.

This grain exhibits a very close resemblance to the pollen of *Loropetalum chinense* (Pl. III, fig 2). The reticulation is so fine as to give a blurred effect at moderate magnification, a character that easily differentiates it from all others described except *Fortunearia*, in which the reticulation is even finer. It differs also from the latter in its much smaller size. This grain has been recorded only in the Bremanoir seam.

Dicoryphe spp. (Pl. III, figs. 13, 14).—Spherical, 3-cleft, slightly oblate, 28–35 μ in diameter, very coarsely reticulate, meshes up to 7 μ across; walls of network relatively high and narrow, somewhat flaccid and easily distorted; furrows shorter than is usual in the furrowed grains of this order, and furrow notches not so prominent when grain is presented in polar view; boundary of furrow and body-wall with zigzag outline in conformity with reticulum pattern, projections on one side opposite to, and, when furrow is closed, fitting closely into recesses in the other.

This grain is distinguished from its allies by the extreme coarseness of its reticulation, and the tendency of the high but narrow walls of the meshwork to be distorted, with the result that the mesh pattern loses its angularity to some extent The furrows, too, are shorter, and do not extend so far polar-wise Thus, when the grain is viewed from the poles (Pl. III, fig 13), its outline is not so deeply notched as is usual in the pollen of the allied genera, and, when viewed from the side, only one furrow is clearly visible (Pl III, fig 14).

A striking feature of the zig-zag furrow margin is the manner in which a projection on one furrow wall faces a re-entrant angle on the other, so that when the furrow closes the two sides fit so exactly that the furrow position may be discernible only with difficulty. A comparison of Pl. III, figs 14, 15, shows the striking correspondence in this respect as well as in mesh character between the fossil and the modern species (*macrophylla*, fig. 15) with which it has been compared. The pollen of *D macrophylla* possibly has shorter furrows than some of the fossil examples, but does not appear to differ in any other respect. Despite the present habitat (Madagascar) of this genus I have felt justified in this identification

The grain is frequent in the Ardslignish seam, but has not been noted in the interbasaltic seams.

Distylium sp. (Pl. III, fig. 16).—Spherical, variable in size, 25–50 μ in diameter, often slightly but regularly indented, or faintly 2- or 3-furrowed, grain shape easily distorted and very variable, exine finely reticulate all over, no evidence of pores.

These grains bear a close resemblance to the pollen of *Distylium racemosum*. The fossil types show a great variation in size, and there is probably more than one species present; but a wide range in size (20–40 μ) is also exhibited by the pollen of *D. racemosum*.

The grains are very common in some preparations of the Ardslignish coal, and at certain levels in the seam are the dominant pollen type. They have not been noted in the Shiaba or Bremanoir samples

Haloragidaceæ.

Cf. *Myriophyllum* sp. (Pl. III, fig. 17).—Body of grain circular in outline, 26 μ in diameter; exine slightly rough; pores 4, surrounded by very much thickened exine and rising abruptly above surface of grain, not symmetrically placed, 3 crowded together, pore apertures long and slit-like, at least one pair converging.

Only one grain of this type has been observed—in the Ardslignish seam. In its details—size, shape, and grouping of the pores—its resemblance to the pollen of species of *Myriophyllum* is such that its close relationship to this genus is considered certain.

It differs from the pollen of the modern species of *Myriophyllum* that I have seen in the extreme prominence of the pores and the very long slit-like apertures. In these two characteristics it somewhat resembles pollen of species of the allied genus *Haloragis*, but, on the other hand, it lacks other distinctive characteristics of *Haloragis* pollen, *e.g.* the thick band of exine which connects the pores spaced at equal intervals on the periphery of the disc-shaped grains.

Wodehouse (1933, p. 516) records from the Green River shales a somewhat similar grain, which he identifies as a species of *Myriophyllum*.

CONCLUSIONS.

A list of the genera recognized and their distribution as between the infra- and interbasaltic seams are set out below.

LIST OF GENERA.

Genera.	Ardslignish (Infrabasaltic).	Shiaba and Bremanoir (Interbasaltic)
Ginkgo	o	×
Pinus	o	×
Cedrus	o	×
Abies	o	×
Podocarpus	o	×
Taxodiinæ Cupressineæ Taxeæ } Genera undifferentiated	o	×
Smilax ?	×	×
Engelhardtia	×	o
Alnus	×	×
Cf Alnus	×	×
Corylus ?	×	×
Planera	o	×
Cf Pteroceltis	×	? ×
Magnolia	×	×
Bucklandia	×	o
Corylopsis	×	×
Fortunearia	×	o
Loropetalum	o	×
Dicoryphe	×	o
Distylium	×	o
Cf Myriophyllum	×	o

As it would be unwise to attempt a detailed review of this flora before a complete survey of the lignites has been effected, only certain features of general interest, which are already apparent, will be briefly alluded to:

(1) *Modern Affinities of the Fossil Flora.*—Of the 20 genera recognized, 7 are now indigenous to East Asia, 1 to North America, 1 to Africa, 1 to East and West Asia, 1 to East Asia and various localities in the southern hemisphere, 1 to East Asia and North America, while 8 are widely distributed throughout the countries of the northern hemisphere The preponderance of the East Asiatic element is even more pronounced when one notes that all but two of the genera have species native in that region.

(2) *Climatic Conditions of the Period.*—Assuming that the modern representatives do not materially differ in habit from their Tertiary ancestors, the climate in Scotland in early Tertiary times was on the whole apparently temperate—probably warm temperate—in character, although it may have been cooler after the onset of vulcanism than in the preceding period.

(3) *Changes in the Vegetation following the Outbreak of Vulcanism* —Inspection of the List of Genera suggests that the composition of the flora probably differed rather profoundly before and after the onset of volcanic activity. In assessing this difference, however, due weight must be accorded the fact that the pollen-content of the lignites is not necessarily a representative sample of the flora of the period any more than the pollen-content of a modern swamp furnishes a record of every species of plant that grew in its neighbourhood.

Nevertheless, such facts as the appearance in abundance in the interbasaltic lignites of the pollen of various conifers and of *Planera*, types which are absent in the infrabasaltic seam, and, conversely, the greatly reduced frequency in the interbasaltic seams of the pollen of *Magnolia* and hamamelid genera so dominant in the infrabasaltic seam cannot be ignored. The conditions of plant growth were doubtless profoundly modified in at least two directions by the volcanic episode:

(a) *Soil Change.*—The basalts (basic rocks) would weather to a soil of quite different composition from that furnished by the sandy expanse of the pre-volcanic period—and therefore attract a different plant-community.

(b) *Topographic Change.*—The pre-volcanic land was probably a terrain of low relief, while the volcanic pile was almost certainly more varied in relief and possibly higher in altitude, or at least dominated by high volcanic cones which might be expected to have a pronounced effect on the rainfall of the area. Significant vegetational changes, then, might well be expected.

SUMMARY

(1) Lignites of Tertiary age from the West of Scotland have been treated for the extraction of spores and pollen grains after the method of Raistrick, and found to possess an abundant and varied pollen-content.

(2) Pollen grains of species of plants belonging to 20 genera have been identified, and are described and figured.

(3) The resemblance of the modern East Asiatic flora to this fossil flora is pointed out, and also the presence in it of certain forms now confined to Africa and America

ACKNOWLEDGMENTS.

The writer wishes to express his gratitude to Dr Bernard Smith, F.R.S., Director of the Geological Survey of Great Britain, for permission to draw upon the Survey collection for a specimen of the Shiaba lignite, and to Mr J Adam Watson, Edinburgh, for a specimen of the lignite from Bremanoir. Special thanks are due to Sir William Wright Smith, Regius Keeper of the Royal Botanic Garden, Edinburgh, for permission to take from the collections in the Herbarium there, samples of pollen of many species of plants desired for comparative purposes, and to Mr Orr, Mr Evans, and Mr Todd, members of his staff, who rendered valuable personal assistance on numerous occasions.

The writer also desires to thank Mr Cotton, Keeper of the Herbarium, Kew, and Mr Ramsbottom, Keeper of the Botanical Department, British Museum, for pollen samples which they supplied; and Professor A. C. Seward, F R S., Cambridge, for many helpful references to literature. The assistance of my colleague, Mr Fisher, in the photographic part of the work is also gratefully recorded.

Lastly, thanks are due to the Carnegie Trust for the loan of a microscope necessary in this investigation and for a grant towards the expenses of the illustrations.

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DESCRIPTION OF PLATES.

(*All figures magnified about* 600 *diameters.*)

PLATE I.

Fig. 1.—*Ginkgo* sp., length 29 μ, showing wavy furrow margin. Slide No. 23 Shiaba (Author's Collection).

Fig. 2.—*Ginkgo* sp., diameter 30 μ, broadly elliptical form. Slide No. 23 Shiaba (Author's Collection).

Fig. 3.—*Pinus* sp., body breadth 70 μ, illustrating marginal ridge, projection of marginal ridge at dorsal root of bladder (left side), and ventral situation of bladders. Slide No. 23 Shiaba (Author's Collection).

Fig. 4.—*Pinus* sp., showing wrinkling of exine of cap (above right-hand bladder). Slide No. 26 Shiaba (Author's Collection).

Fig. 5.—*Pinus* sp., dorsal view, diameter of cap 70 μ. Slide No. 1 Bremanoir (Author's Collection).

Fig. 6.—*Magnolia* sp., length 48 μ, showing longitudinal furrow, and the granular-reticulate nature of the exine. Slide No. 22 Ardslignish (Author's Collection).

Fig 7.—*Magnolia* sp, length 70 μ, showing particularly the character of the furrow. Slide No 18 Ardslignish (Author's Collection)

Fig 8.—*Magnolia* sp, length 43 μ, showing furrow and smooth nature of exine Slide No 25 Shiaba (Author's Collection).

Fig 9.—*Magnolia* sp., length 62 μ, an example of the larger-sized, rough-textured species. Slide No. 6 Ardslignish (Author's Collection).

PLATE II

Fig 1.—*Pinus* sp., ventral view, bladders touching but not collapsed, marginal ridge seen on outside of left bladder. Slide No. 1 Bremanoir (Author's Collection)

Fig. 2.—*Cedrus* sp., showing lateral position of bladders, etc. Compare with *Pinus* (Pl. I, fig. 5). Slide No. 15 Shiaba (Author's Collection).

Fig 3.—*Cedrus* sp, diameter about 70 μ, ventral view illustrating typical contracted condition The bladders are moulded upon the ventral surface, but on account of their lateral position do not completely cover it Also the margin of the dorsal surface (cap), visible through the bladder membrane, falls well inside outer margins of bladders. Compare with *Pinus* (Pl II, fig. 1). Slide No. 9 Shiaba (Author's Collection).

Fig 4.—*Abies* sp, complete length 125 μ. Slide No. 2 Shiaba (Author's Collection).

Fig. 5.—*Podocarpus* sp, dorsal view, illustrating the relatively enormous bladders and small body (35 μ by 45 μ). Slide No. 20 Shiaba (Author's Collection).

Fig. 6.—Cf. *Alnus* sp., large size (48 μ) from infrabasaltic seam Slide No 17 Ardslignish (Author's Collection).

Fig. 7.—Cf. *Alnus* sp., large size (44 μ) from interbasaltic seam. The character of the pores is clearly displayed. Slide No. 19 Shiaba (Author's Collection).

Fig. 8.—*Alnus* sp., 5-pored (30 μ) grain. The curved thickenings between the pores are prominent. Slide No. 19 Shiaba (Author's Collection).

Fig. 9.—*Alnus* sp., 4-pored grain. Slide No. 1 Shiaba (Author's Collection).

Fig. 10.—*Planera* sp., typical 4-pored (25 μ) grain, showing the pattern of the exine, the ridging caused by the curved thickenings, and the fleck in the centre of one pore Slide No. 12 Shiaba (Author's Collection).

Fig. 11.—*Planera* sp., 5-pored grain, showing oval aperture of pores, and divergent thickened bands. Note also the shorter length of one (right) side of pentagon. Slide No. 12 Shiaba (Author's Collection).

PLATE III.

Fig. 1.—*Loropetalum* sp., 17½ μ in diameter. Slide No 1 Bremanoir (Author's Collection).

Fig. 2.—*Loropetalum chinense* Oliver, modern pollen.

Fig. 3.—*Bucklandia* sp., typical (34 μ) grain in polar view. Compare with fig. 4. Slide No. 19 Ardslignish (Author's Collection).

Fig. 4.—Polar view of the modern *Bucklandia populnea* R. Br.

Fig. 5—Cf *Pteroceltis* sp., 31 μ by 27 μ. Note fleck in pore at right-hand bottom corner. Slide No. 21 Ardslignish (Author's Collection)

Fig. 6—*Smilax*? sp. (28 μ) The patchy distribution of spinose projections is notable. Slide No. 15 Shiaba (Author's Collection)

Fig 7.—*Bucklandia* sp., side view The highly oblate shape (typical of many hamamelids) and the elongated meshes in equatorial region are evident. There is possibly a trace of the furrow-membrane preserved in this grain. Slide No 2 Ardslignish (Author's Collection).

Fig. 8—Polar view of a modern hamamelid (*Corylopsis sinensis* Hemsl), about 28 μ in diameter.

Fig. 9.—*Corylopsis* sp. Note the equidimensional character of the mesh. Slide No. 2 Ardslignish (Author's Collection).

Fig. 10.—*Corylopsis* sp., 35 μ. Slide No. 3 Shiaba (Author's Collection).

Fig 11.—*Fortunearia* sp, 26 μ, illustrating the finely reticulated exine, and traces of black flecks on margin of pore membrane. Slide No 2 Ardslignish (Author's Collection).

Fig 12.—Polar view of the modern *Fortunearia sinensis* Rehder and Wilson. The black flecks surrounding the pore area in one furrow are distinct

Fig. 13.—*Dicoryphe* sp. (28 μ), polar view illustrating the very coarsely reticulated exine, and typical furrow shape. Slide No. 19 Ardslignish (Author's Collection).

Fig. 14—*Dicoryphe* sp, side view, showing typical furrow outline. Compare with fig. 15. Slide No. 22 Ardslignish (Author's Collection)

Fig. 15.—Side view, showing one furrow, of the modern *Dicoryphe macrophylla* Baill.

Fig. 16.—*Distylium* sp., large (50 μ) grain showing typical vague indentations and finely reticulated exine Slide No. 19 Ardslignish (Author's Collection).

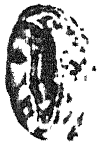
Fig 17—Cf. *Myriophyllum* sp., body 26 μ. Note 3 pores along base of triangle, the slit-like aperture of middle pore and the circular base of its exine thickening faintly visible. Slide No 17 Ardslignish (Author's Collection).

Fig. 18.—*Engelhardtia* sp. (18½ μ), illustrating the clear spaces beneath the pores and the finely granulated central area Slide No. 5 Ardslignish (Author's Collection).

Fig. 19.—Pollen grain, length 36 μ, of an undifferentiated coniferous species of the groups Taxodiinæ, Cupressineæ, Taxeæ. Slide No 3 Shiaba (Author's Collection).

Fig. 20.—*Corylus*? sp. (35 μ). Slide No. 2 Shiaba (Author's Collection).

(*Issued separately August* 5, 1936.)



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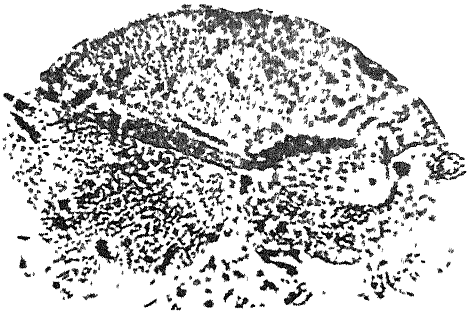

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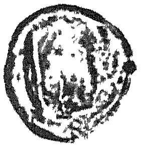

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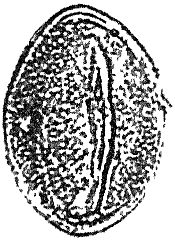

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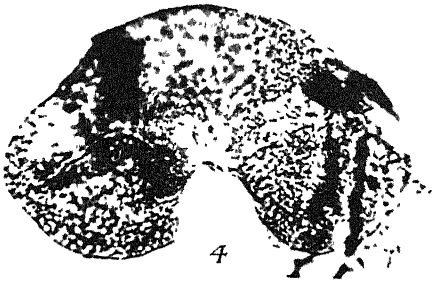

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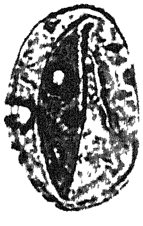

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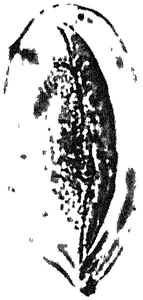

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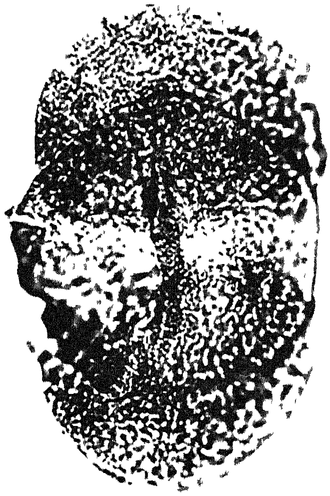

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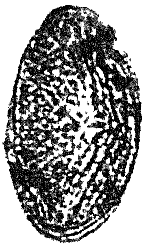

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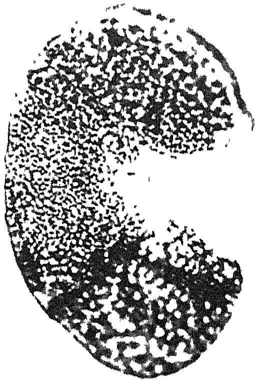

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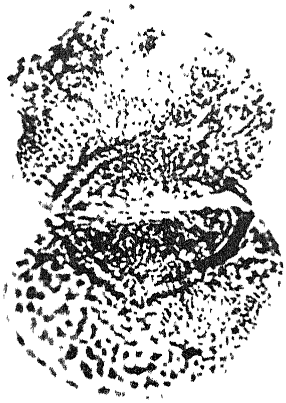

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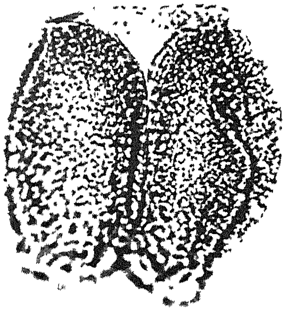

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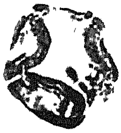

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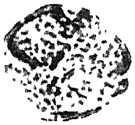

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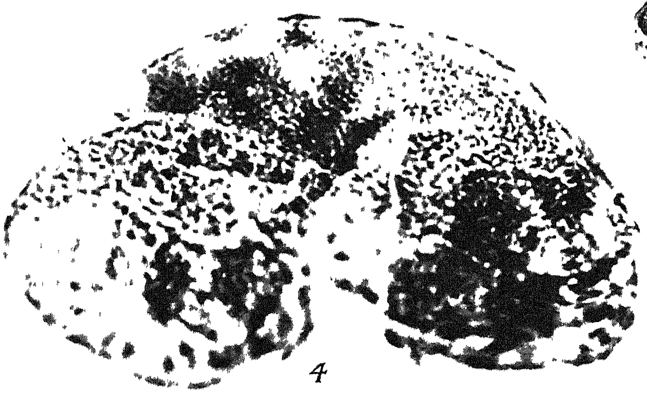

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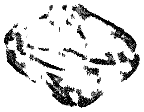

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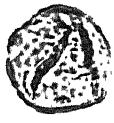

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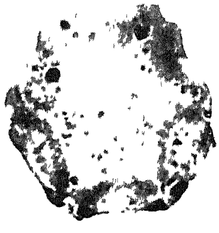

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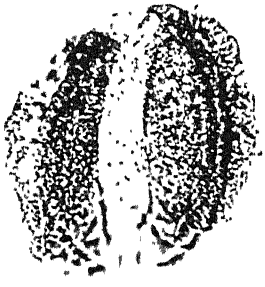

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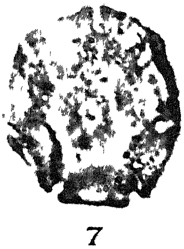

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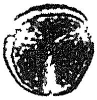
PLATE II.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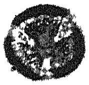
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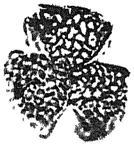
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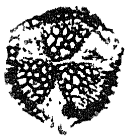
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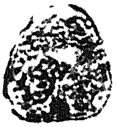
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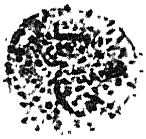
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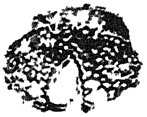
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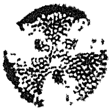
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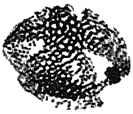
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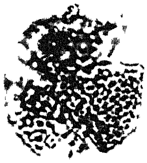
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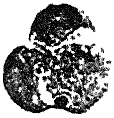
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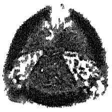
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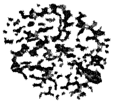
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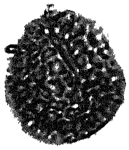
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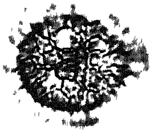
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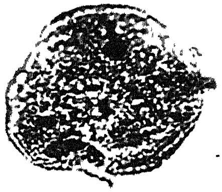
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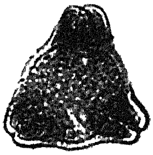
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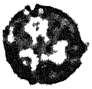
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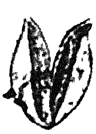
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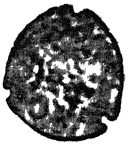
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VIII.—On the Suppression of Tangled in *Drosophila pseudo-obscura*.

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(With Two Plates and One Text-figure.)

(MS. received April 29, 1936. Read June 1, 1936.)

INTRODUCTION.

THE results of the interaction of any two genes cannot yet be foretold with any degree of accuracy. This is another way of saying that our knowledge concerning the problem of the gene and development is still very meagre even in its grosser details. The work of Schultz (1935) and others has shed much light on the reasons for such phenomena as the disproportionate modification of vermilion by purple (Bridges and Morgan, 1919), and for many other observed effects following the combination of eye-colour mutants, but explanations for similar phenomena in other parts of the body, for instance, the non-additive action of bristle modifiers found by Plunkett (1926), and the interaction of the wing and bristle mutants of Lebedeff (1935), are of a somewhat more speculative nature. The accumulating data on these problems do not yet allow the empirical method to be dispensed with in seeking the answer to the question: What will happen when these two mutants are put together? From another point of view, the interaction of two genes, when it can be observed, is a useful method of approach in the investigation of genetic control in development, and has been used by various workers such as Csik (1934), Goldschmidt (1935), Dunn and Coyne (1935).

Simple summation effects for increasing doses of a given gene or its allelomorphs appear to be the rule (for general discussion of this subject and relevant literature see Mohr, 1932). Plunkett (1926), Stern (1929*a*), Dobzhansky (1929), and Gowen (1933) have all found that the effects of genes affecting bristle size are more or less proportional to their number. Non-allelomorphic genes have also been found to act in this way. Kikkawa (1934) has reported that in *D. virilis* the genes Confluent and plexus which, like the gene tangled of this paper, are characterised by extra plexate venation, produce when combined more extra venation than either alone. An additive effect in the opposite direction occurs when the mutants radius incompletus and venæ transversæ incompletæ of *D. funebris* are combined (Timoféeff-Ressovsky, 1927) A third type of combination of mutant genes affecting the amount of venation, namely, one increasing and the other decreasing venation, has been studied in *D. pseudo-obscura*, and this has also shown an additive effect. The investigation which is reported here arose from an attempt to discover the nature of two "inhibitors" of the mutation tangled. These "inhibitors" proved to be mutant genes causing gaps in the venation so that the investigation became a study of the interaction of several genes as well as of the nature of "inhibitors" and "suppressors" Up to the present, only the qualitative aspects of the matter have been considered, but there is clearly scope for quantitative work on the suppression of tangled.

DESCRIPTION OF MUTANTS AND STOCKS USED

(1) *Tangled*.—The mutant tangled (*tg*), which arose as a spontaneous mutation, is a fourth chromosome recessive located at one end of the known linkage group (Crew and Lamy, 1935). No visible effects can be detected in the heterozygous flies, but in the homozygotes there is considerable disturbance of the venation in the region of the posterior cross-vein and near the ends of the second and third longitudinal veins. The disturbance takes the form of extra development of vein-material which gives rise to large plexate patches and additional cross-veins. As a rule, the wings are also blistered, particularly at the posterior cross-vein, and may occasionally contain bubbles of liquid The best idea of the phenotypic appearance of this mutant can be obtained by reference to Pl. I, figs. 2, 3, which show typical tangled wings. There is some variation in its manifestation which may be slighter than that shown. The wings are not symmetrical, but they always agree in showing approximately the same degree of disturbance. The wing shape may be narrower than normal at the tip, and one or both wings curled upward. There is little if any difference between the sexes The two plexate areas are more or less equal in size in the majority of tangled wings. The one at the posterior cross-vein is always considerable, although sometimes obscured by blistering The other is more variable in extent. This point is of interest in connection with the effects of introducing the following mutant genes affecting vein-length.

(2) *Short*$_4$—On the fourth autosome also, the mutant short$_4$ (s_4)

(Crew and Lamy, *loc. cit.*) has been located 50 or more units from tangled. It appears to be a complete recessive, and when homozygous causes a shortening of the fourth and fifth longitudinal veins, and nearly always the elimination of the posterior cross-vein (see Pl I, fig. 4) There is not a great deal of variation in expression in present stocks, but occasionally a fly with the fourth longitudinal complete but weak at the tip will be found.

(3) *Short.*—Another short (*s*) (Lancefield, 1922), located on the X-chromosome at about 180, also causes a shortening of the fourth and fifth longitudinals, but to a much slighter extent (Pl. I, fig. 5) At room temperature many of the males may show little more than a weakening at the tip of the fifth longitudinal. All the females and most of the males, however, show a short fourth longitudinal and an unaffected posterior cross-vein The phenotypic expression of this short is therefore much less than that of the autosomal short$_4$

(4) *Tilt*—Of greater variability than either of the preceding shorts is the mutant tilt (*tt*) (Crew and Lamy, *loc cit*) which is located close to short at about 182 on the X-chromosome. In some tilt stocks only a few of the females may show it at all as a slight gap at the end of the third longitudinal The rest are wild-type Nearly all males in such a stock show the mutation and have rather larger gaps in the vein, not necessarily at the end of it (Pl I, fig. 6). The fourth longitudinal may also be slightly shortened. In other stocks the expression is much greater The shortening of the third vein appears in all females, and frequently gaps in the fourth and fifth, and occasionally in the second longitudinals occur. These gaps are all fairly short. In the males, the third vein is missing almost to the anterior cross-vein, the fourth and fifth to the posterior cross-vein, and the second for a short distance at the tip (Pl. I, fig. 7)

(5) *Shadow-tangled*—The character shadow-tangled (*shtg*), which acts as an allelomorph of tangled, appears usually as a little extra venation between the distal ends of the second and third longitudinal veins, but this may sometimes be reduced to a faint blurring, or a dot, or may not show at all (Pl I, fig. 8). It appeared in the F_2 of a mating of one of the original tangled females to miniature short$_4$ jaunty males, as shadow-tangled short$_4$ jaunty flies, and upon a long inbred stock raised from these, part of the present investigation was carried out.

(6) *Tangled Inhibitor.*—One of the F_1 females from which the above F_2 was raised had been outcrossed and had produced a son with all the longitudinal veins short. He must have been also heterozygous tangled, for, on outcrossing, he gave rise to an F_2 with tangled flies. These tangled flies were mass cultured for several generations, and on examination

they were then found to include many males which were not tangled, some of them were wild-type and some had a gap in the third or fourth longitudinal vein. Several pair matings now showed that certain females were heterozygous for a "tangled inhibitor" which was sex-linked and recessive By selection, a stock was obtained in which the females were wild-type or slightly tangled, and in which the males had sometimes gaps in the veins but were usually wild-type. These males, although never tangled, gave only tangled offspring when outcrossed to stock tangled females.

The stock of "tangled inhibitor" used in these experiments shows a certain amount of variation (Pls. I, II, figs 9–13). The males may resemble a weak type of tilt, or may be quite wild-type, or they may have a posterior cross-vein—that is, three-quarters to half the normal size, and perhaps slightly askew. This is the only kind of variation on the tangled side of normality which is ever seen. The females, on the other hand, are often wild-type, but rarely have any gap in the venation, and they may show varying degrees of slight tangling in both the usual places. The greatest amount of tangling is at the posterior cross-vein, and as the wild-type is approached the last traces are always seen here.

EXPERIMENTAL RESULTS

(a) *The Suppression of Tangled by Short$_4$.*—It has already been reported (Donald, 1936) that linkage experiments with a stock of shadow-tangled short$_4$ jaunty ($shtg\ s_4\ j$) flies gave rise to the belief that shadow-tangled was the outcome of the interaction of tangled and short$_4$. This has now been proved by the synthesis of tangled short$_4$ jaunty flies which are shadow-tangled and breed true when mated to stock shadow-tangled flies. The experimental results are as follows.—

(1) The first crosses made were $\frac{shtg\ j\ s_4}{+\ +\ +}$ ♀♀ × $shtg\ j\ s_4$ ♂♂ and the results published in connection with the linkage data obtained from them. They showed that most of the shadow-tangled flies were at the same time short$_4$ and that the remaining tangled flies were not of the usual stock type as expected, but were intermediate between shadow-tangled and tangled; that is to say, there were clear extra veins in both the usual regions of the wing, but the tangling was "open" and without the large plexate areas; there were no blisters, and the wings were flat (Pl. II, fig. 14). About half the total number of tangled flies (see Table I) was comprised of this type.

TABLE I.—OFFSPRING OF $\frac{shtg\ j\ s_4}{+\ +\ +}$ ♀♀ × $shtg\ j\ s_4$ ♂♂

Class.	Genotype	First brood ♂♂	First brood ♀♀	Second brood. ♂♂	Second brood. ♀♀
+	$\frac{tg\ j\ s_4}{+\ +\ +}$	189	235	65	81
$shtg\ j\ s_4$	$\frac{tg\ j\ s_4}{tg\ j\ s_4}$	122	153	72	87
$shtg$ $intg$	$\frac{tg\ j\ s_4}{tg + +}$	28 100	2 160	11 48	. 70
$j\ s_4$	$\frac{tg\ j\ s_4}{+ j\ s_4}$	128	149	97	93
j	$\frac{tg\ j\ s_4}{+\ j\ +}$	15	26	15	8
$shtg\ s_4$	$\frac{tg\ j\ s_4}{tg + s_4}$	17	29	11	12
s_4	$\frac{tg\ j\ s_4}{+\ + s_4}$	28	37	15	15
$intg\ j$ $shtg\ j$	$\frac{tg\ j\ s_4}{tg\ j\ +}$	19 6	43 4	18 1	24 3
		652	838	353	393

		♂♂.	♀♀
Total	+	284	350
,,	$shtg\ s_4$	222	281
,,	s_4	268	294
,,	$intg$ $shtg$	231	306
,,	flies	1005	1231

Disregarding jaunty, the observed proportions shown on the right of Table I are reasonably close to the ratio 1 : 1 : 1 : 1 expected if the s_4 *shtg* flies are genetically $s_4\ tg$, and the *shtg* and *intg* flies are genetically $tg\ \frac{s_4}{+}$ and arise from crossing over between the s_4 and *tg* genes which show free recombination. Subsequent work verified this interpretation. The dominant phenotypic effect thus produced by the heterozygous short$_4$ gene was considerable. The reduced form of tangling was easily distinguished from the stock type of tangled, but not always from shadow-tangled which it overlapped frequently, especially among the males. Table I shows that quite a large number of males were classified as shadow-tangled without short$_4$. Since the latter always appears when homozygous even in the presence of homozygous tangled, these males must have been genotypically the same as those classified as intermediate tangled (*intg*), namely, $\frac{s_4\ tg}{+\ tg}$.

A corresponding phenotypic effect of tangled in $\frac{s_4\ tg}{s_4\ +}$ flies seems probable and the impression that the veins were more shortened in such flies than in s_4 *tg* flies is supported by comparison of stock cultures of short$_4$ with and without tangled, but the effect, if real, is not very marked.

(2) The next step was to produce shadow-tangled short$_4$ flies by starting with the cross tangled by short$_4$. Jaunty was present in this cross, but has been disregarded. The F$_2$'s from this cross were not raised to completion after it was observed that the expected classes occurred according to the following scheme:—

Cross	Eggs.	Sperm.	Phenotype.
$\frac{\quad s_4}{tg\quad} \times \frac{\quad s_4}{tg\quad}$	s_4	s_4 *tg*	s_4 +
	tg	s_4 *tg*	+ *tg*
	s_4 *tg*	s_4 *tg*	s_4 *intg*
	+	s_4 *tg*	+ +

At the same time as these F$_2$'s were made, two F$_1$ females were back-crossed to stock shadow-tangled short$_4$ males. The results of these matings are given in Table II.

TABLE II —$\frac{\quad s_4}{tg\quad}$ ♀♀ × *shtg* s_4 ♂♂ (2 CULTURES)

Class	♂♂.	♀♀.	Genotype.
s_4	30	37	$\frac{tg\ s_4}{+\ s_4}$
intg	28	29	$\frac{tg\ s_4}{tg\ +}$
shtg s_4	12	20	$\frac{tg\ s_4}{tg\ s_4}$
+	24	42	$\frac{tg\ s_4}{+\ +}$

As expected, there were no typical tangled flies. The shadow-tangled short$_4$ cross-over class was exactly the same as stock, and it was considered satisfactorily established that the shadow-tangled character was due to the partial or complete suppression of tangled by short$_4$.

(b) *The Suppression of Tangled by Short.*—In order to find out if short would have the same suppressing effect on tangled as short$_4$, short females were mated to tangled males. F$_1$ females mated to tangled males produced offspring that would be expected if the two shorts had much the same effect (Table III).

TABLE III —$\frac{s}{+}\ \frac{tg}{+}$ ♀♀ × *tg* ♂♂.

Class.	♂♂	♀♀.	Genotype.	Class.	♂♂.	♀♀.	Genotype.
tg	29	53	*tg/tg*	*s shtg*	25	..	*s tg*
+	51	114	*tg*/+, *s*/+ *tg*/+ (♀)	*shtg*	1	20	*s*/+ *tg* (♀)
s	42	.	*s tg*/+, *s tg*	*intg*	1	9	*s*/+ *tg* (♀)

Although not equal in size, the classes of males are otherwise as expected (Pl. II, figs 15–17). Apparently a male genetically short tangled may sometimes not show short, and a male genetically tangled may show a rather slight type of tangled Among the females there are again too few of the reduced tangled type (some were probably classified as wild-type), but it seems clear that the presence of heterozygous short has caused the females to be classified mostly as shadow-tangled and not intermediate-tangled as in the experiments with short$_4$ Short has therefore a greater suppressing effect in the heterozygous condition than short$_4$ In fact, many of the $\frac{s}{+}$ *tg* ♀♀ showed very slight traces of tangled, traces as slight as those produced by homozygous short$_4$ tangled females.

(c) *The Suppression of Tangled by Tilt.*—The inference that the "tangled inhibitor" might be the mutant tilt was now an easy one, and has been proved correct by the process of making synthetic inhibited-tangled flies and mating them to stock flies Tilt females were first crossed to tangled males All sons should have been tilt, but a few appeared wild-type This might be attributed to a converse effect, namely, the suppression of tilt by tangled, but the evidence from tilt stocks and from tilt flies heterozygous for tangled is inadequate to show if tilt males which are heterozygous for tangled approach the wild-type more often than tilt males which are not. F_1 females were then chosen to back-cross to tangled males as they should produce inhibited-tangled sons if tilt were the inhibitor, and would at the same time show how heterozygous tilt compared in effect on tangled with the heterozygous shorts.

TABLE IV.—OFFSPRING OF $\frac{tt}{+}\ \frac{tg}{+}$ ♀♀ × *tg* ♂♂ (5 CULTURES).

Class	♂♂	♀♀	Genotype.	Class.	♂♂.	♀♀.	Genotype
tt	68		*tt tg*/+	*tt intg*	4		*tt tg*
tg	44	203	*tg*, *tt*/+ *tg* (♀)	*tt* short C_2	5	.	*tt tg*
+	122	222	*tt tg*, *tg*/+ (♂♂) *tt*/+ *tg*/+, *tg*/+ (♀♀)	short C_2	15	.	*tt tg*
intg	52	2	*tg*, *tt tg*				

Total ♂♂ 310 ♀♀ 427

Unfortunately, these results are not very easy to interpret. There can be no doubt that tilt has inhibited tangled to a certain extent, but to what extent is not clear. One quarter of the total males should have been tilt-tangled, but actually less than one-thirtieth of the total number (9 in 310) appeared. At first sight it seems probable that they are included in the class intermediate-tangled, but there are two objections to this. One is that the intermediate-tangled class differed from that in previous experiments in being difficult to distinguish from tangled rather than from shadow-tangled, so that while it probably contains some tilt-tangled males, the majority will be tangled only Another objection is that tangled would show a viability much poorer than that of the wild-type, whereas they are consistently about equal in other experiments. The conclusion must be, therefore, that the wild-type class also contains many tilt-tangled males. The classes called "short C_2" had no other sign of tangled than a shortening of the posterior cross-vein like that occurring in the tangled-inhibitor stock, and on testing proved to be of the constitution tilt-tangled. The four males appearing as *tt intg* are obviously tilt-tangled, and show that tilt can be present in a wing with a good deal of tangling.

It is clear that there is a good deal of variation in the appearance of the tilt-tangled males of this experiment (Pl II, figs. 18, 19). They varied from a fairly good tilt, through wild-type, short posterior cross-vein with and without tilt, to slight and intermediate tangling, but no typical tangled was found to be at the same time tilt. In stocks of the tangled inhibitor the tangling in males is further reduced than this, being practically eliminated. The best explanation that can be offered is that selection in the "tangled inhibitor" stock has accumulated modifiers of tilt which are known to exist, and which increase its expression and at the same time its suppressing effect on tangled. Further evidence on this point is forthcoming from experiments with tilt flies that carried lanceolate (*ll*, 1–140), which is associated with an intensifying effect on tilt (see below). Table IV shows also that tangled females which are heterozygous for tilt show no appreciable difference from those which are not. In this respect, therefore, tilt differs from the shorts in being practically recessive in its suppressing action.

As anticipated, the various grades of tangling found in this experiment are not exactly the same as those in the previous experiments. For this reason, the term shadow-tangled has been dropped as it is properly used only to describe those flies which have a slight degree of tangling at the distal ends of the second and third longitudinals and none at the posterior cross-vein This is the type produced by the combination of the shorts with tangled. With tilt the tangling disappears last at the posterior cross-vein and is greater there than in the anterior region at all grades of expression. This implies that the shorts have a different "intensity distribution" from that of tilt.

In order to determine whether or not a more extreme type of tilt would have a greater suppressing effect, lanceolate-tilt females were mated to tangled males, and their daughters back-crossed to tangled males. The offspring of five such F_1 females are classified in Table V (Pl. II, fig. 20).

TABLE V.—OFFSPRING OF $\frac{ll\ tt\ tg}{+\ +\ +}$ ♀♀ × *tg* ♂♂

Class	♂♂	♀♀	Genotype of Males	Class.	♂♂.	♀♀	Genotype of Males.
ll tt	50	.	*ll tt tg*/+	*tg*	45	177	*tg*
				intg	20	3	*tg*, *tt tg*
+	67	183	*tg*/+, *tt tg*	*shtg*	10	4	*tt tg*
ll	33		*ll tg*/+	*tt shtg*	3	.	*tt tg*
tt	20	.	*tt tg*/+, *tt tg*	*ll shtg*	6	.	*ll tt tg*
				ll tt shtg	13	.	*ll tt tg*
				ll tg	10		*ll tg*

Total ♂♂ 277 ♀♀ 367

Among these flies the intermediate-tangled are again to be regarded as tangled and not tilt-tangled. There may be one or two tilt-tangled flies which appeared as intermediate-tangled, but since tangled and intermediate-tangled were difficult to distinguish, whereas shadow-tangled was of a very slight type and easily separated from intermediate-tangled, it is considered safe to regard the intermediate-tangled flies as tangled. Disregarding lanceolate, the totals become:

	♂♂	
tt	70	(68)
tg (*intg*)	75	(96)
+	100	(122)
tt tg	32	(24)
	277	(310)

The figures in brackets refer to the corresponding totals from the previous experiment in which there was no lanceolate. The conclusion drawn from these figures is that many of the flies classified as wild-type in both experiments were actually tilt-tangled, and that some of the intermediate-tangled flies of the previous experiment with tilt were also tilt-tangled This means, if correct, that the use of the lanceolate-tilt flies has brought about a relative decrease in the amount of tangling, this, in turn, supports the suggestion made above that the absence of even slight tangling in the males of "tangled inhibitor" stocks is due to the accumulation of genetic intensifiers of tilt

Females of the constitution $\frac{ll\ \ tt}{+\ \ +}$ *tg* were now mated to stock males carrying tangled and the inhibitor. If tilt is actually the inhibitor, their sons should be half tangled and half tilt, shadow-tangled, or wild-type, and their daughters half tangled and half like the females of the inhibitor stock.

Table VI.—$\frac{ll\ tt}{+\ +}$ *tg* ♀♀ × Stock *tg* Inhibitor ♂♂ (3 Cultures)

Class	♂♂	♀♀	Class	♂♂	♀♀
tg	16	37	+	6	34
intg	..	15	*ll*	7	
ll tg	1	..	*ll tt*	22	
			tt	12	

Total ♂♂ 64 ♀♀ 86

The total number of flies is unexpectedly small owing to the unrealised presence of two mutants of low viability—knobby eyes and gouty legs—in the parents, but even though the actual proportions of the various classes has little or no significance, the appearance of wild-type sons and daughters proves the identity of the tangled inhibitor with tilt.

As a check, several males classified in Table VI as wild-type, but showing only a short C_2, were mated to tangled females. These produced only tangled offspring. It is perhaps worth noting that as tilt occurred as a mutation among flies with and without tangled, its rôle as inhibitor was obscured.

Discussion.

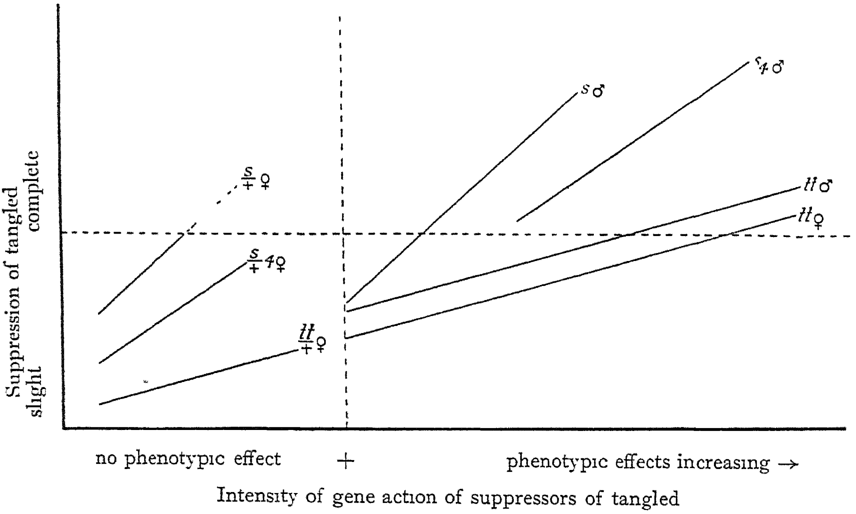
The results obtained may be explained on the following hypothesis. When the mutant tangled is introduced into the wild-type fly the usual process of vein-formation is altered so that excess vein material is formed in definite regions of the wing. Whether this means that the necessary internal environment for the action of tangled exists only near the ends of the wing, or whether the usual processes are less easily disturbed in the proximal parts, need not be decided for present purposes. The opposite kind of effect is produced by the vein-shortening mutants which may be supposed to inhibit the process of vein-formation or to remove its products in the distal parts of the wing. If these two opposed influences are at work simultaneously and independently, the result will depend on whichever was the stronger. Underlying the term "strong" may be actually rate or period of activity, moment of initiation, or combinations of these and other factors.

It has been found that the suppression of tangled is not confined to those parts of the wing which normally exhibit the interrupted venation. It is therefore necessary to suppose that short, short$_4$, and tilt exert a suppressing effect on the tangled venation in regions of the wing where they are never able to suppress normal vein-formation. A more exact study of the variation in the intensity of action of a gene over various parts of its field is possible with bristle modifiers, and the work of Sturtevant and Schultz (1931) and Rokizky (1931) shows that a gene may have a field of action which varies irregularly about a maximum. The actual degree of suppression of tangled achieved by these mutants is therefore regarded as varying in different parts of the wing. Since the shorts, which shorten the fourth and fifth longitudinal veins, have a greater suppressing effect at the posterior cross-vein than they do at the end of the third and second longitudinals, their field of influence must fade away from a maximum which is presumably at or near the point where they produce phenotypically visible effects in the absence of tangled. Tilt causes gaps to appear in the third longitudinal veins, and may therefore be considered to have a maximum effect in that region That its effect may extend farther on both sides is shown both by the appearance of gaps in the other longitudinals in flies with good manifestation of the character, but also in the reduction in tangling on both sides of the third longitudinal. The considerable variation found among flies genetically tangled and short, short$_4$, or tilt, seems to indicate that the variability of the individual mutants is retained in their combinations. If this is so, then it is necessary to postulate that their mutual suppression may be compatible with a measure of independent activity; that is to say, neither mutant in a given combination actually suppresses the working of the other, but a quantitative adjustment occurs in accordance with the opposed nature of their end-products.

Another source of variability in the degree of suppression is the actual dosage of the mutants. No obvious evidence that tangled when heterozygous can reduce the manifestation of any of its homozygous suppressors has been found, but the reverse is clearly true, at least for the shorts. Heterozygous short or short$_4$ females which are at the same time homozygous tangled show an unmistakable reduction of tangling which may approach complete suppression. Nevertheless, tilt, which may have the most extensive phenotypic effects, does not seem able to produce much suppression in the heterozygous female. These facts provide further evidence (Timoféeff-Ressovsky, 1934) that genes which are completely recessive or dominant are probably very scarce. By introducing tangled into the genotype, short is able to produce a strongly dominant phenotypic effect, short$_4$ a weaker one, and tilt possibly a very weak one.

A certain amount of sexual dimorphism has also been found in the results, particularly in connection with tilt. It is noticeable, for instance, that the single dose of the sex-linked tilt in the male produces a definitely greater suppression than a double dose in the female. This is in keeping with the sex-dimorphism of stock tilt.

On the basis of the foregoing considerations, the following diagram has been constructed to illustrate graphically the preceding hypothesis to explain the phenotypic effects observed in the various combinations of tangled with the vein-shortening mutants.



The diagram represents the hypothetical relation between the "intensity" of gene action of the suppressors as measured by the degree of phenotypic expression in the absence of tangled, and their suppressing effect on tangling in the region of the posterior cross-vein. Suppression, however, is mutual, and though occasional flies may show a degree of tangling in this region together with a slight shortening of the veins, the suppressors will in general begin to manifest themselves in the presence of tangled at the point where their curves cut the abscissa, indicating complete suppression of tangled On the basis of the comparison of *tt tg* and *ll tt tg* flies, the sex-dimorphism, and the observation that the extreme types of manifestation of the suppressor are not found in the presence of tangled which appears to be incompatible with even a moderate manifestation of the suppressors, it is assumed that the degree of suppression is proportional to the phenotypic expression without tangled. This may be expressed in other words as an additive relation between tangled and its suppressors and is represented by a straight line, the ends of which are determined by the range of variation in phenotypic effect The appropriate lines have been drawn so that the abscissa for complete suppression of tangled divides them into parts with the same ratio as that of tangled to non-tangled flies in a population carrying both tangled and the suppressor.

In the area to the left of the ordinate marked + the intensity of gene action of the heterozygous suppressors is below the threshold for phenotypic manifestation. The end points of the curves here were estimated from the observed variation in suppression and the lines drawn parallel with the corresponding ones in the right-hand section of the diagram.

Since the range of variation of tangled itself is not negligible, the curves refer to one point in that range, and all the observed variation in tangling is ascribed only to the effect of the suppressors.

According to a recent definition which states that "the early term 'inhibitor' may be used in the general sense of a genetic agent of unspecified nature which prevents another genetic factor from showing its effects," and that the term "specific suppressor" is to be applied when "a given mutant character is inhibited by the action of a second mutant gene, the double mutant type and the second mutant type both appearing like unmutated wild-type" (Bridges, 1932*a*), the mutants short, short$_4$, and tilt are doubtless to be classed as "inhibitors" of tangled

The various specific suppressors which have been recorded affect a fairly wide range of mutants The original suppressors of vermilion and sable have been succeeded by others affecting such widely different mutants as purple (Stern, 1929*b*; Bridges, 1932*b*), black (Plough, 1928), scute, deltex, and cut (Bridges, 1932*a*). The nature of these suppressors is at present uncertain. Recent work by Bridges (1935, 1936) on duplications in salivary gland chromosomes has renewed the possibility that the specific suppressors may be associated with duplications involving the locus of the gene suppressed, but whether they can all be explained in this way is still doubtful Indeed, the foregoing experimental results might be construed as evidence that some suppressors not of this nature may be found

It might be argued with some justification that, from one point of view, short was an inhibitor of tangled, and from another that it was a suppressor of tangled. For instance, in so far as short produces a shortening of veins, it must be regarded in the light of the above definition as an inhibitor, but by the same token it is a suppressor because it prevents the appearance of tangled at the distal end of the second longitudinal vein where it never appears itself Short might also be regarded as a suppressor when it is heterozygous in the female, but not when it is homozygous. The same considerations apply with appropriate variations to the other mutants dealt with here.

A comparison of tilt with short shows that a gene with a slight phenotypic effect is not necessarily one with a negligible internal effect Certainly mutants with slight phenotypic effects have often the least detrimental influence on the viability of the flies which possess them, but this is by no means the rule. Further, Timoféeff-Ressovsky (1934) has drawn attention to the fact that the proportion of dominants among the mutants with poor manifestation is about the same as that among those with good manifestation. It is not surprising, then, to find genes which have no phenotypic manifestation at all suppressing others which have. It would easily be possible to construct a series of mutant genes showing a gradation in visible effect from 0–100 per cent., but the position of any gene in this series would be no indication of its ability to prevent the expressions of another gene with an opposed characteristic. Thus specific suppressors could be modifiers without visible effect, but passing gradually into those with visible effects, namely, inhibitors. It seems unlikely that such specific suppressors could be distinguished by the time at which they acted to produce their suppression. It is clear that whether they act at the same time, before, or after the gene of which the phenotypic effects are suppressed, the same latitude must apply to inhibitors as well.

Without prejudice to the actual processes by which somatic effects are produced, the difference between the "inhibition" of tangled and the "suppression" of, say, black might be considered as connected with the fact that the intensity of gene action in the "Wirkungsfeld" of short, short$_4$, or tilt varies so that the point of complete suppression is not attained under all conditions in all parts where tangled appears, whereas the black suppressor passes the threshold of suppression in all parts of the body.

The fact that specific suppressors prevent the appearance of all the characteristics of "pleiotropic" genes indicates that even if the latter were not really pleiotropic (as suggested by Kamschilow, 1935) they are the direct and original cause of the development of all the associated characteristics which could not appear if there were any gene (*e g* a suppressor) preventing the primary reactions of the first mutant gene, or removing the products of them. The mutual suppression of Notch^8 and Abrupt-X (Nazerenko, 1930) seems to come into a different category The pleiotropic effects of Notch^8 are probably due to a deficiency of a number of genes, and therefore would not arise from one initial deviation from the normal processes of development. A single gene "suppressing" dissimilar processes involving several genes is less likely to be found than one "suppressing" a single process, involving one mutant gene

Schultz and Bridges (1932) have pointed out that suppressors might be used as examples of modifying genes which Fisher (1930, 1932) has postulated for maintaining the dominance of the wild-type The fact that all the specific suppressors shown to be genes are recessive would indicate on Fisher's theory that they had not originated as neutral or advantageous mutations, but as unfavourable ones which had themselves been modified by selection in the direction of the wild-type. This means that they could have begun as facultative inhibitors. In a population where two mutants are regularly appearing which have opposed end-products there will be as the result of selection the tendency for each to become progressively more effective as a suppressor, and at the same time to have its own phenotypic effects reduced to a minimum. This is more or less the situation now presented by the specific suppressors. They have reached the stage of being phenotypically invisible and at the same time more or less complete inhibitors of another recurring mutation. Put another way, some specific suppressors might be at the last stage in the conversion of a gene complex into a condition in which it no longer reacts to a particular mutation.

Certain of the steps in such a mode of evolution of a specific suppressor might be illustrated as follows:—

(1) *Tilt.*—This mutant has a considerable and obviously disadvantageous phenotypic effect when homozygous, and the genetic environment which enables it to suppress tangled to the greatest extent is the one which increases its own manifestation.

(2) *Short$_4$.*—Suppresses tangled more than tilt, and is still accompanied by a fairly considerable phenotypic effect.

(3) *Specific Suppressor of Hairy-wing.*—Has reached the stage of complete suppression, but its own somatic effects are far from negligible

(4) *Short.*—A slight phenotypic effect overlapping wild-type and a further increase in ability to suppress tangled.

(5) *The Recessive Modifier of Frizzle in Fowls* (Hutt, 1932; Landauer, 1933)—Reduces the expression of Frizzle, and has apparently no somatic effects of its own

(6) *Specific Suppressor of Vermilion.*—If there are actually no somatic effects, this mutant would have reached the last stage in which suppression is complete, and somatic effects modified to the point of identity with wild-type

These stages are obviously not comparable *inter se* owing to the differences in the nature of the mutants suppressed and of their suppressors, and it is not intended to convey that such a series has any considerable evolutionary significance But if specific suppressors may have arisen in this way, it is difficult to distinguish them arbitrarily from those which have arisen *de novo* in the final stages.

The fact that a specific suppressor was recessive in its action would offer no difficulties to the comprehensive theory of dominance proposed by East (1935). On his view, these would be recessive because they caused a restriction in the physiological processes in which they were involved From the standpoint of Fisher's theory, this fact would be surprising. It would be expected that if a mutant had an advantageous or neutral effect it would show at least partial dominance. This is true of the inhibitors of tangled, which therefore might be considered as examples of mutants with both advantageous and disadvantageous effects of which the former have retained their dominance and the latter have been modified in the direction of recessivity. Such mutants would unite the two conditions discussed by Fisher (1930, p. 65), namely, the dominance of advantageous mutations and the recessiveness of disadvantageous mutations.

SUMMARY

1. Two different types of suppression of the character tangled in *Drosophila pseudo-obscura* have been found to be due to genes causing interrupted venation.

2 The experimental results suggest that tangled interacts in an additive way with its suppressors, and that modifiers tending to increase phenotypic manifestation act in the same way.

3. Three genes which have been found to suppress tangled have recessive phenotypic effects, but at least two have dominant suppressing effects, and all have suppressing effects in regions of the wing in which they are not themselves manifested.

4 The inverse relation which was found between the phenotypic expression and the degree of suppression effected by these three mutants suggests that there may be no clear line of demarcation between mutants coming into the categories of "specific suppressors" and "inhibitors"

ACKNOWLEDGMENTS

The author wishes to acknowledge his indebtedness to Professor F. A. E Crew for his generous hospitality and assistance, and to Miss R. Lamy for the use of stocks and for help in the execution of the experiments which have been made upon them.

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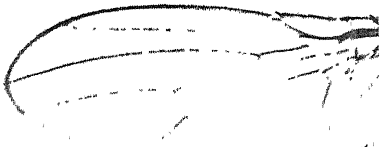

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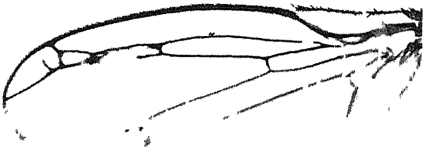

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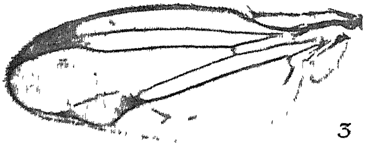

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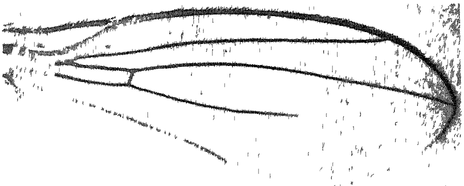

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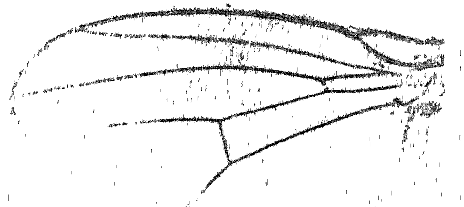

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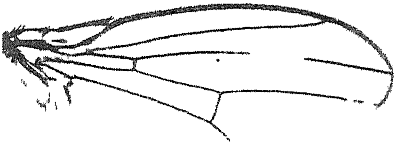

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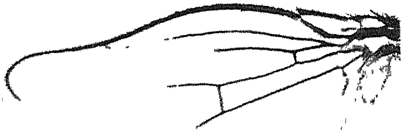

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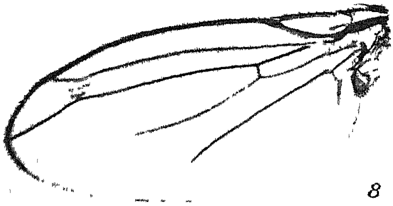

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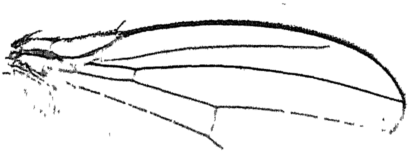

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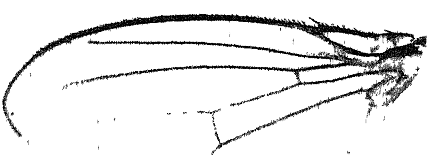

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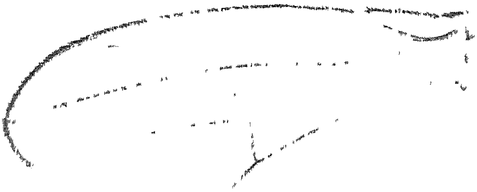

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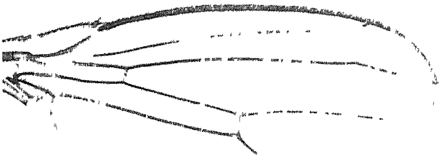

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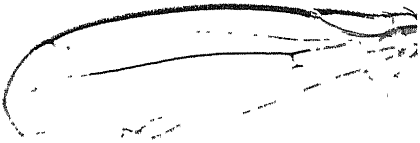

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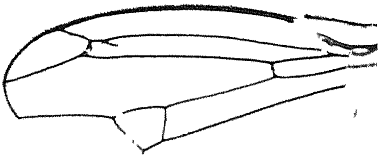

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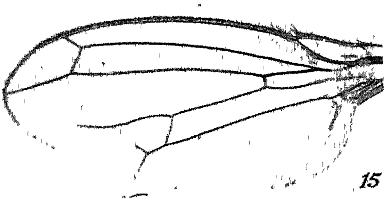

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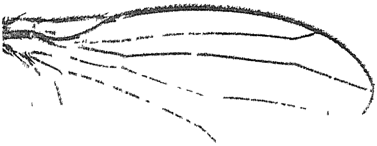

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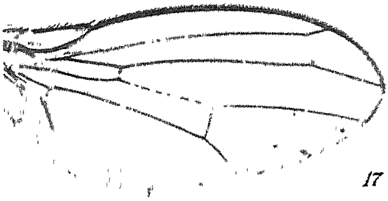

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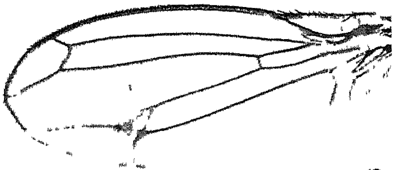

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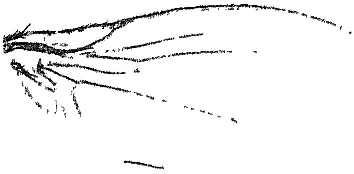

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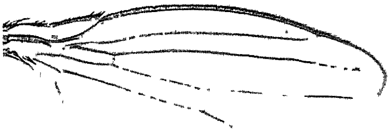

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DESCRIPTION OF PLATES.

PLATE I.

Fig. 1. Wild-type.
,, 2 Tangled.
,, 3 ,,
,, 4. Short_4.
,, 5 Short.
,, 6. Tilt (♂).
,, 7. Lanceolate-tilt (♂).
,, 8. Short_4 tangled.
,, 9 Inhibited tangled (♂).
,, 10. ,, ,, (♂)

PLATE II.

Fig. 11 Inhibited tangled (♀).
,, 12. ,, ,, (♀).
,, 13. ,, ,, (♀).
,, 14. Intermediate tangled $\left(\frac{s_4}{+}\, tg\ ♀\right)$.
,, 15. Short tangled (♂).
,, 16. ,, ,, (♂).
,, 17 ,, ,, (♂).
,, 18. Tilt tangled (♂).
,, 19. ,, ,, (♂).
,, 20 Lanceolate tilt tangled (♂).

(*Issued separately August* 5, 1936)

IX.—The Dynamics of the Formation of Cone-sheets, Ring-dykes, and Caldron-subsidences.

By **E. M. Anderson**, M A , D Sc., F.G.S. (With Nine Text-figures.)

(MS received April 7, 1936 Read June 1, 1936)

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5	Origin of the Magma Reservoirs	154
6	Summary . . .	156
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1 INTRODUCTION.

A SYSTEM of intrusions of the type now known as cone-sheets was first definitely recognized and mapped by Harker in the Cuillin district of Skye, and was described by him in the Geological Survey memoir issued in 1904. The members of the complex were simply designated "inclined sheets," but they incline inwards towards a common centre from north, west, south, and south-east. When the Scottish Survey undertook the mapping of Mull, in the years preceding the war, two further series of centrally inclined intrusions were soon recognized. Owing to the number of these sheets, and consequent liability to confusion, individual members can seldom be traced very far, or shown to have arcuate curvature. But the systems as a whole sweep round centres in continuous curves. In one case the curve was originally closed, though it has been somewhat interrupted by later intrusions. In the other case the main body of sheets only forms a horse-shoe. Nevertheless the term cone-sheets, which was here applied to these intrusions by Professor Bailey, is extremely appropriate

The intrusions forming the horse-shoe are the earlier in date. They dip towards their centre at angles of about 45°, and average maybe 30 ft. in thickness. They include two main petrological types, but the petrology of these and the other igneous rocks here dealt with has already been fully described, and will only be incidentally referred to in this paper

The width of the horse-shoe is about ten miles. The cone-sheets which belong to it are intersected by those of the other system, which dip somewhat more steeply towards a centre lying two or three miles farther to the north-west. The earlier sheets are very numerous, while the later are so abundant as to be often contiguous, leaving only remnants of the igneous rocks into which they were originally intruded (Bailey, 1924).

Well-developed cone-sheet systems also occur in Ardnamurchan (see fig. 1), where they have been described by Richey (1930), and a few sheets have been found in Rum. There is also a series round the Carlingford plutonic centre in Northern Ireland, where they had been mapped by Traill, but their identity as a cone-sheet system was first recognized by Richey. This does not complete the list, as, except for Arran, cone-sheets seem to occur round all the important British Tertiary centres.

Outside of Britain the cone-sheet principle of intrusion has hardly as yet been recognized Recently, however, what appears to be a series of very thick cone-sheets has been found by Bain in the Kudaru Hills of Nigeria (1934).

A point of some importance with regard to the mechanism of these intrusions was noticed by Harker in Skye, where the inner members of the complex were found to be dipping at steeper angles than the outer. This is also the case, at least to a limited extent, with regard to the later sheets in Mull. The rule has, in addition, been found by Richey to apply to the cone-sheets connected with "Centre 2" in Ardnamurchan, where, however, the inner and steeper sheets are later in date than the outer.

In sharp contrast to these centrally inclined intrusions are the more or less vertical ring-dykes. Individual ring-dykes were already known, for instance, that partly encircling the Glencoe caldron-subsidence. The occurrence of a numerous system of concentric ring-dykes was, however, a possibility first realized in Mull, and it was during the survey of Mull that the term ring-dyke was introduced by Bailey and first applied In Mull there are two centres, which are certainly connected with the same underground reservoirs which gave rise to the two sets of cone-sheets. Round the earlier or Beinn Chaisgidle centre there is a complex system of ring-dykes, although it must be noted that in using this word one refers more often than not to cases in which the ring is incomplete. The arcs of a circle round which individual ring-dykes can be traced are, however, much larger than in the case of individual cone-sheets. The complex as a whole is about four miles in diameter. Fully half of the area which it occupies is made up of ring-dykes, although there are "screens" of pre-existing rock between. In the case of one of the innermost members the ring is only a mile in diameter, and nearly complete.

The Beinn Chaisgidle centre was the same as that of the early cone-sheets. The ring-dykes were later than these, but mainly earlier than the cone-sheets of Centre 2. The latter centre is surrounded by one nearly

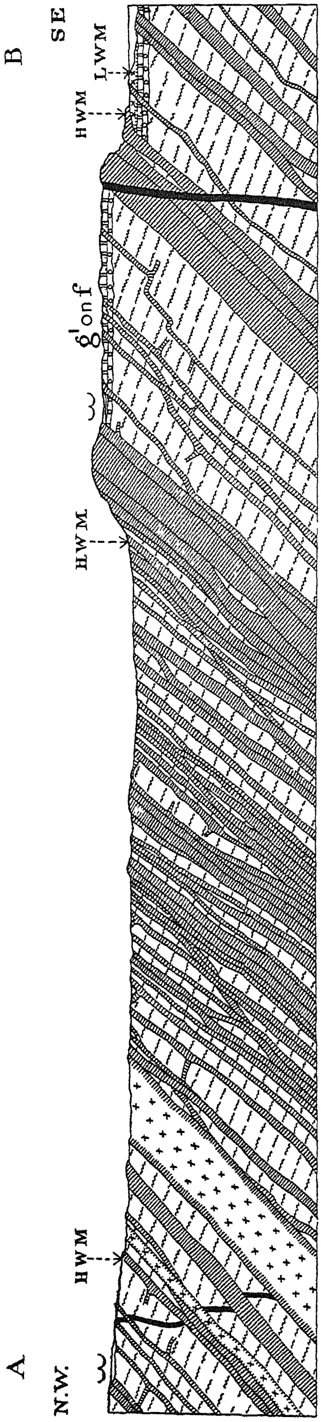

Fig. 1.—Section showing cone-sheets (outer set of Centre 2) along shore south of Kilchoan Ardnamurchan
Reproduced from "The Geology of Ardnamurchan, etc," *Mem Geol Surv*, fig 24, p 175

complete ring-dyke, the Loch Bà Felsite. The curve is, however, not strictly circular, although it has bilateral symmetry about a north-western axis. This dyke is later than the cone-sheets of its own centre, and in contrast to them has an outward inclination which varies from 45° to 70° or 80°, as judged from its columnar structure It followed a pre-existing fissure which was itself a complete ring This fissure was a fault with central downthrow, and the amount of throw is estimated at one point to be not less than 3000 ft.

The Glen More ring-dyke is classed as the outermost member of the suite connected with the earlier or Beinn Chaisgidle centre. Its form, however, suggests that it came under the influence of both centres, and it is of special interest also in another respect. At one point this nearly vertical intrusion splits, sending off a horizontal apophysis, which, however, soon abruptly turns upwards, and itself becomes vertical. "It seems fair to claim these exposures as furnishing an ocular demonstration of the simultaneous employment of two parallel fissures by the ascending magma of a ring-dyke" (Bailey, 1924, p. 314).

In Ardnamurchan Richey has shown the presence of two series of ring-dykes (fig 2). The earlier of these was developed round Centre 2, which has already been noticed, partly during the interval of time which separated the intrusion of the two series of cone-sheets The later ring-dykes are developed round another centre, and are extremely perfect They border one another with hardly any remnants or "screens" of the intruded rock, and the order of intrusion was towards the centre. The rings thus became successively smaller in diameter, and the centre is occupied by a boss

The structure of this centre has influenced the contours of the ground, forming a striking case of "lunar topography."

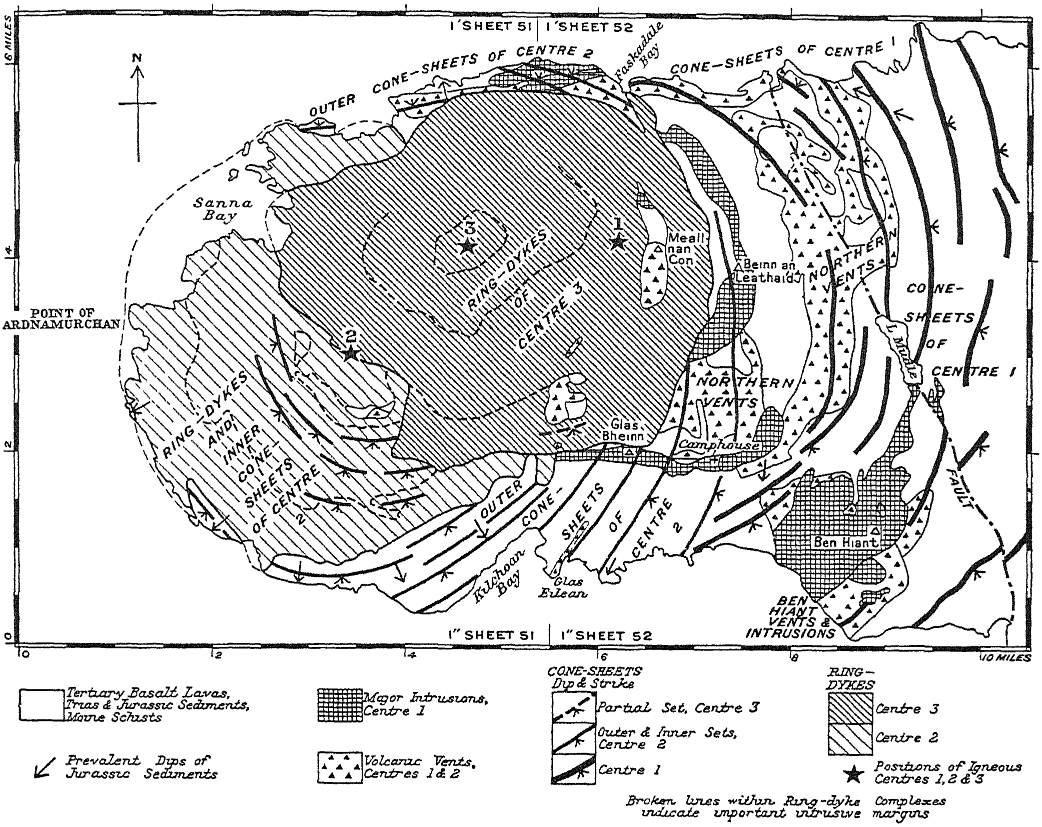
1'SHEET 51
1'SHEET 52
N
OUTER CONE-SHEETS OF CENTRE 2
Faskadale Bay
CONE-SHEETS OF CENTRE 1
Sanna Bay
POINT OF ARDNAMURCHAN
3
1
2
RING-DYKES OF CENTRE 3
Meall nan Con
Beinn an Leathaid
NORTHERN VENTS
CONE-SHEETS OF CENTRE 1
L. Mudle
NORTHERN VENTS
Glas Bheinn
Camphouse
RING-DYKES AND INNER CONE-SHEETS OF CENTRE 2
OUTER CONE-SHEETS OF CENTRE 2
Kilchoan Bay
Glas Eilean
Ben Hiant
FAULT
BEN HIANT VENTS & INTRUSIONS
1" SHEET 51
1" SHEET 52
0
2
4
6
8
10 MILES
6 MILES
Tertiary Basalt Lavas, Trias & Jurassic Sediments, Moine Schists
Major Intrusions, Centre 1
CONE-SHEETS Dip & Strike
Partial Set, Centre 3
Outer & Inner Sets, Centre 2
Centre 1
RING-DYKES
Centre 3
Centre 2
Prevalent Dips of Jurassic Sediments
Volcanic Vents, Centres 1 & 2
Positions of Igneous Centres 1, 2 & 3
Broken lines within Ring-dyke Complexes indicate important intrusive margins

FIG 2.—Ring-dyke complexes and cone-sheets of Ardnamurchan
Reproduced from "The Geology of Ardnamurchan, etc ," *Mem. Geol Surv.*, Pl. II, p 71.

A fine instance of a Tertiary ring-dyke complex has also been described by Richey round Slieve Gullion in Ireland (1932). Here, as in the case of the Loch Bà Felsite, individual rings seem to have an outward inclination, which can be ascertained at a few places to be about 70°. The Carlingford plutonic complex consists largely of a ring of gabbro (eucrite), which may be a ring-dyke, surrounding a central stock of granite. Both these intrusions are obviously developed from the centre which has been responsible for the formation of the cone-sheets.

Outside of Britain the best-known ring-dyke complex is probably that of the Pilansberg in South Africa. This has been recently described by Shand, who, however, explains the structure as being that of a modified laccolite (1928). Similar intrusive complexes have also been found in New Hampshire (Kingsley, 1931), in Utah, and elsewhere, and it seems probable that as geological investigation proceeds their number will be extended.

The type instance of a dissected caldron-subsidence is that of Glencoe, though similar structures may at present be in course of formation under, for instance, Santorin, and other examples of the wider type of volcanic vents. In Glencoe, however, dissection has shown how an annular fault gave rise to a wide crater of subsidence, and itself probably formed the channel through which a succession of lavas were poured out. These occupied the subsidence, and possibly overflowed. The fault dips outwards on the northern margin at angles of 50° to 70°, but this is balanced by an inward dip on the southern side It seems probable in fact that when formed the marginal fracture was in most places nearly vertical, and that there has been subsequent tilting.

The classical account of the Glencoe subsidence which was given by Clough, Maufe, and Bailey (1909) contains also another suggestion which is probably of great importance in the tectonics of intrusion. The Glencoe fault came to the surface, but it is possible for a ring-fracture to form in depth without reaching the surface, but extending downwards to some liquid substratum, and thus leaving the rock within its circuit hanging as it were, and dependent partly on the tensile strength of the layers above. Under these circumstances the stresses which gave rise to the fault-ring would be profoundly modified, and a cross-fracture might be formed, under which the block as a whole might sink. The more or less cylindrical space left vacant might be filled by an ascending magma, for which, as before, the ring-fracture would act as a channel A granite stock might well be formed in this manner, and the hollow space it might be supposed to fill was named in this paper a "subterranean cauldron." The adjacent Cruachan granite, which belongs to the same period of igneous activity as the Glencoe subsidence, was shown by the writers to be steep-sided, while in the case of its inner core, the Starav granite, the vertical nature of the margin had already been made out. These cases are not, however, exceptional. The very word "stock" implies a steep-sided intrusion. Often, however, the sides have an outward inclination, and are not precisely vertical, as shown in the writers' diagram (reproduced as fig. 3).

A structure which is probably very closely allied to that of ring-dykes has been brought to light by Richey's study of the relations of the different granites constituting the Mourne Mountains in Ireland (Richey, 1928).

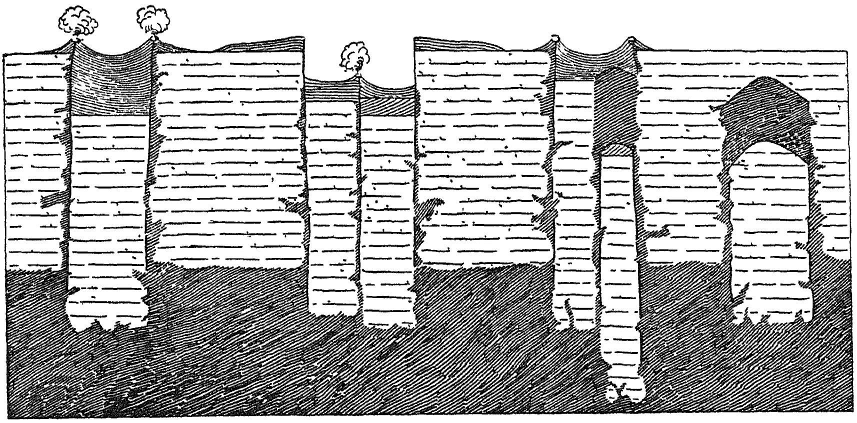

FIG 3.—Diagram illustrating subaerial and subterranean caldron-subsidences accompanied by volcanic and plutonic accumulations of igneous rock.

Reproduced from "The Geology of Ben Nevis and Glen Coe," *Mem Geol Surv.*, fig 24, p. 126.

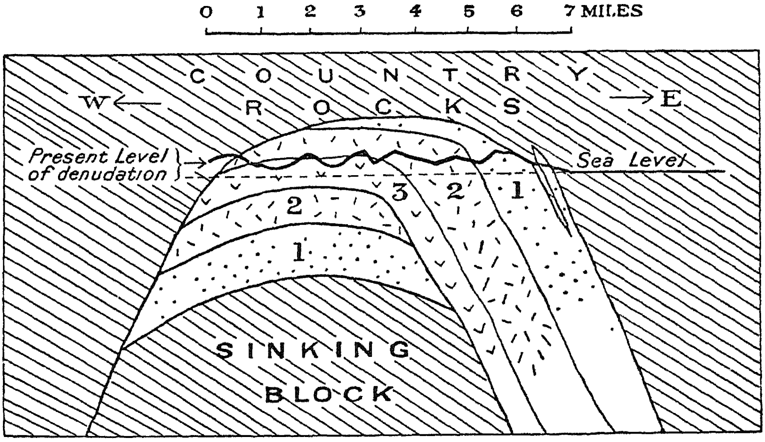

FIG 4.—Diagram illustrating supposed mode of intrusion of granites of Eastern Mourne Mountains.

Reproduced from "The Structural Relations of the Mourne Granites" (J. E Richey), *Quart. Journ. Geol Soc.*, vol. lxxxiii, 1928, fig. 7, p. 685

There are here three successive intrusions. The first two partially surround the third in a manner which is more or less arcuate. The boundary surfaces of all three are in part vertical, or somewhat outwardly inclined.

When a certain height is reached, however, these steeply dipping surfaces bend over abruptly to a position which is nearly horizontal (fig. 4). Thus granite No. 2 overlies as well as surrounds granite No 3, and No. 1 has the same relation to No. 2. There is a suggestion also that the outer boundary of granite No 1 behaves in the same way Thus granite No. 1 may be regarded as having filled a subterranean caldron-subsidence This, however, was only the first stage of intrusion After No 1 had consolidated, a second caldron-subsidence must have formed within it, with lesser radius and a lower roof. This process was again repeated, leading to the intrusion of No. 3. The flat roof must in each case mark different dynamical conditions from the nearly vertical boundary, in the manner which has been explained above In other words, it must have been a cross-fracture.

It is well known that granite stocks are often margined by rings of igneous rock of somewhat different composition from the interiors. The successive formation of caldrons with diminishing radius may therefore have been a common process, but there is no other known case in which the difference in roof level of these caldrons can so well be seen.

The outward inclination of a granite margin is finely illustrated in the case of the Castle Peak stock, British Columbia, which has been figured by Daly (1912, figs. 36 to 40). The roof of the intrusion seems, however, in this instance, to have been formed in a method different from cross-fracture This is well shown in two of the diagrams (figs. 36, 40). In fig. 40 the margin is seen to bend round, not abruptly, but very gradually upwards from an angle of 75° to one of only 20°.

A further but perhaps less frequent type of central intrusion is the formation of radial dykes. Harker believed that a system of vertical dykes exists in Skye, radiate to, and not extending to more than about three miles from a centre within the Cuillins This is presumably the same centre which supplied the cone-sheets. The evidence is not, however, very definite A more certain example occurs in the Spanish Peaks district of the United States. The dykes here radiate from a stock or stocks of diorite; a map of the relationship has been reproduced by Daly (1933, fig. 22).

The data about the different types of central intrusion and fracture may now be summed up as follows. Cone-sheets and ring-dykes often surround the same centre, but in some cases only the one or only the other may be present. Where both are present the ring-dykes may succeed the cone-sheets in time, but this rule is not invariable, and sometimes there is alternation. Cone-sheets have an inward dip, which tends to increase as the centre of intrusion is approached. The exact inclination of ring-dykes cannot very often be seen, but sometimes they are vertical, while in other cases there is a distinct outward tendency It should be noted too that ring-dykes are as a rule much the more massive. The majority of cone-sheets vary perhaps between 10 ft and 50 ft. in thickness (3 to 15 m.), and ring-dykes from 50 yards to over a mile (say 50 to 2000 m.) The fracture bordering the Glencoe caldron-subsidence was originally vertical The outer margins of granite stocks may, it is suggested, in most cases be explained as subterranean ring-fractures, and these, like some ring-dykes, have an outward tendency.

Such then is a brief résumé of the structures which it is the object of the present paper to explain. For further particulars reference should be made to the various private papers and Survey memoirs which have been quoted, and more especially to Richey's paper on "Tertiary Ring Structures in Britain" (1932), of which I have made the fullest use I wish also to express my debt for personal help received from Dr Richey. Scotland and Northern Ireland are in the meantime the type areas for this class of structure, and Dr Richey's part in their investigation has been a very large one

A preliminary attempt at explanation was made by myself in the Geological Survey memoir on the Tertiary Geology of Mull, etc. (Bailey, 1924, pp. 11, 12). The theory there put forward was essentially the same as that which is now developed, except that it was non-mathematical in its treatment It is therefore convenient to quote this attempted explanation almost in full.

"A dynamical theory of the formation of cone-sheets and ring-dykes may be developed on the following lines. One may disregard the probable existence of a broad lava-cone and other irregularities, and suppose that, speaking very generally, the surface of the ground, at the periods at which these features were developed, was horizontal. Underneath this horizontal surface, at a depth of several miles, was a magma reservoir. Its shape may have been roughly that of a paraboloid of revolution.

"One may suppose that at first the magma had a specific gravity equal to that of the surrounding rocks, and that it was under a pressure just high enough to have raised it to surface level if there had been an outlet. Overlooking the fact that the rocks themselves were not quite homogeneous, one may further suppose that they were under the same horizontal and vertical pressures at every point as would obtain in a liquid with the same surface and the same specific gravity. While such was the case no fractures could arise.

"*Cone-sheets.*—If, however, these conditions were modified by an increase of pressure in the magma-basin, the pressure-system in the crust would have superimposed on it a system of tensions, acting across surfaces which near the basin were roughly conical The fine firm lines in fig. 1" (fig. 5 of present paper) "are intended to show the intersection of these surfaces with the plane of the diagram. A superimposed system of pressures would also act across surfaces which cut the former orthogonally and are indicated in section by the fine broken lines The superimposed tensions, together with the increased pressure of the magma, might cause a series of fractures to develop, along which the magma would intrude. The opening fractures would follow the fine firm lines of fig 1, and thus may have originated the cone-sheets of Mull. It will be seen from the diagram that if the surface were denuded to a certain depth the cone-sheets exposed might be expected to be steeper in the central parts of the area of intrusion. This is actually the case.

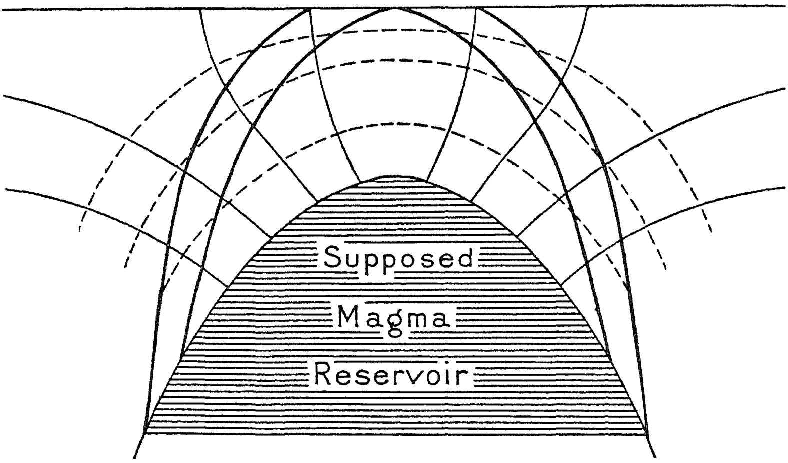

FIG. 5.—Stress-diagram to show supposed mode of formation of cone-sheets and ring-dykes. For explanation of lines, see text.

Reproduced from "The Tertiary and Post-Tertiary Geology of Mull, etc.," *Mem. Geol. Surv*, fig. 1, p 12

"*Ring-dykes.*—If the conditions were reversed and the pressure of the magma fell below that which was at first assumed, the original 'hydrostatic' pressure in the crust would be modified in a different way. Superimposed pressures would act across the surfaces whose trace is shown by the fine firm lines of fig. 1, and superimposed tensions across those which are indicated by the broken lines. It seems likely that, in this case, surfaces of fracture would originate inclined at an angle to the surfaces across which there were maximum superimposed tensions, as in the case of normal faults (Anderson, 1905). The angle may have been about 20°

or 30°. Such surfaces of fracture correspond, not to tension-cracks, but, in theory at least, more nearly to planes of maximum shearing-stress. They deviate from the directions across which this stress is an absolute maximum owing to certain considerations of friction. An attempt to show the trace of such surfaces, taking the angle mentioned as about 25°, has been made in the diagram It can easily be seen that the theory explains the tendency to an outward slope, which is perhaps a feature of ring-dykes. If the fractures formed curves that were closed in cross-section, the rock inside them might tend to become detached and to sink down into the magma The gap between the subsiding mass and the stationary walls would widen with the subsidence, and this perhaps explains, in part, the greater width of ring-dykes when compared with cone-sheets It is uncertain whether, in the majority of cases, the ring-fractures continued upwards to the surface of the ground. When they did so they must have given rise to circular depressions. Such a subsidence of the surface is known to have happened at Glencoe, and it appears to have happened in Mull in the case of the supposed south-eastern caldera."

There is bound, of course, to be a tendency to tangential tension round any convexly bounded enclosure containing fluid under pressure. There must also be a tendency near it to a radial pressure. If the enclosure contains only vacancy and there is pressure in the surrounding solid, the conditions are more or less reversed. There will then be a tangential pressure along the walls with vanishing normal pressure; one has only to consider the example of an arch. The mathematical formulation of the cone-sheet problem proved, however, somewhat difficult, and it would be practically impossible to calculate the stresses corresponding either to a parabolodial or any other arbitrarily chosen shape of the underground caldron. Two solutions of the equations of equilibrium have nevertheless been found, satisfying the surface conditions and giving displacements which might be caused by the presence of a magma basin It is probably of little consequence that the different possible surfaces of pressure which will produce these displacements are none of them paraboloidal. In one of the solutions the surfaces open downwards, while the directions of stress seem to fit the facts of central intrusion fairly well

2. MATHEMATICAL SOLUTIONS.

Use is made, in the first place, of two well-known systems of strain which may occur in a homogeneous, isotropic, and infinite solid. The first of these is that which surrounds a "Centre of Dilatation." In this the displacement is radially outward from a given point and inversely proportional to the square of the distance. This is expressed by

$$u = A\frac{x}{r^3}; \qquad v = A\frac{y}{r^3}; \qquad w = A\frac{z}{r^3}, \qquad . \qquad (1)$$

where u, v, and w are the displacements parallel to x, y, and z These equations represent the strain which would be produced in such a solid if a tiny sphere at the origin were to expand equally all round.

The other strain was investigated by Lord Kelvin, and was named by Boussinesq the "premier type simple d'intégrales" (1885) It is given by a set of relations such as

$$u = B\frac{xz}{r^3}; \qquad v = B\frac{yz}{r^3}, \qquad w = B\left(\frac{z^2}{r^3} + \frac{\lambda + 3\mu}{\lambda + \mu}\frac{1}{r}\right), \qquad (2)$$

where λ and μ are Lamé's constants. These displacements would be the type produced if an imaginary finger could be inserted into a homogeneous solid, from some fourth dimension, at the origin, and were to exert on it a force in the direction of z. Some simple term for the stress system appears to be rather necessary, and it is suggested that it may be appropriately named the "point-push." (1) and (2) both satisfy the equations of equilibrium

$$(\lambda + \mu)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Delta + \mu\nabla^2(u, v, w) = 0,$$

where $\Delta = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$.

These solutions refer, however, to an infinite solid. To apply them to the case of central intrusions one must find the effect (1) of a centre of dilatation, and (2) of an upward point-push in the "semi-infinite" solid which is otherwise unlimited, but bounded upwards by a horizontal plane. This plane may be denoted by $z = 0$, z being measured downwards, while x and y are horizontal. The solutions must have the property that near some point 0, 0, a the other terms which they contain must become negligible in comparison with those which denote the point dilatation or the point-push. They must satisfy the equations of equilibrium, and in addition the components of traction must vanish across the plane $z = 0$ I shall use the symbols P_{yz}, etc., to denote the tractions, with the usual convention that P_{lm} is the force per unit area, in the m direction, acting on the l face of the element dx, dy, dz from the side on which l is greatest. P_{xx}, P_{yy}, and P_{zz} will then be positive when they denote tensions and negative when they denote pressures.

To obtain the solution for the case of the point dilatation one must revert for the meantime to the supposition that one is dealing with an infinite solid. One must then suppose that the dilatation at o, o, a is balanced by a similar dilatation at o, o, $-a$. The resulting displacements are

$$u_1 = A\left(\frac{x}{r_1^3} + \frac{x}{r_2^3}\right); \qquad v_1 = A\left(\frac{y}{r_1^3} + \frac{y}{r_2^3}\right); \qquad w_1 = A\left(\frac{z-a}{r_1^3} + \frac{z+a}{r_2^3}\right),$$

where r_1 and r_2 are the distances from the points o, o, a and o, o, $-a$ respectively It can easily be shown that where $z=0$ the tractions P_{zx} and P_{zy} vanish, while

$$P_{zz} = 2\mu\frac{\partial w_1}{\partial z} + \lambda\Delta = 4\mu A\left(\frac{1}{r^3} - \frac{3a^2}{r^5}\right),$$

where r denotes either r_1 or r_2.

One may endeavour next to obtain a displacement u_2, v_2, w_2, which shall satisfy the field equations, and give rise to the same values of P_{zz} across the plane $z=0$, where it likewise makes $P_{zx} = P_{zy} = 0$. If such a displacement can be found, containing no singularities where z is positive, and vanishing at infinity, then by subtracting its components from u_1, v_1, w_1 we shall arrive at a resultant strain which is a solution of the problem.

This result may be attained by using a theorem of Hertz, which may be stated as follows (1881). If to the plane $z=0$ which bounds as before a semi-infinite solid we apply any general pressure $p=f(x, y)$ without associated shearing stress, then if P be the electric potential of a density $\frac{p}{4\pi}$ of electricity at every point on the plane, the displacements which are caused by the pressure may be derived from this potential. The quantity Π is first defined as follows:—

$$\Pi = -\frac{zP}{\mu} + \frac{1}{\lambda+\mu}\left[\int_z^\infty P\,dz - J\right], \qquad . \quad . \quad (3)$$

where J is a constant so chosen as to make Π finite. We then have

$$u_2 = \frac{\partial\Pi}{\partial x}; \qquad v_2 = \frac{\partial\Pi}{\partial y}; \qquad w_2 = \frac{\partial\Pi}{\partial z} + \frac{2(\lambda+2\mu)}{\lambda+\mu}P \quad . \qquad (4)$$

One has next to consider what is the potential of a distribution $\frac{-\mu A}{\pi}\left(\frac{1}{r^3} - \frac{3a^2}{r^5}\right)$ of electricity, over an infinite plane, where r is the distance from a point whose perpendicular distance from the plane is a. This can be found by the method of electric images. Where z is positive the potential will in fact be that of an electric dipole, of direction OZ and strength $2\mu A$, situated at o, o, $-a$. Its value is thus

$$\frac{2\mu A(z+a)}{r_2^3}.$$

We have therefore

$$\Pi = 2\mathrm{A}\left[\frac{-z(z+a)}{r_2^3} + \frac{\mu}{\lambda+\mu}\frac{1}{r_2}\right],$$

from which we obtain

$$u_2 = 2\mathrm{A}\left\{\frac{3xz(z+a)}{r_2^5} - \frac{\mu}{\lambda+\mu}\frac{x}{r_2^3}\right\},$$

$$v_2 = 2\mathrm{A}\left\{\frac{3yz(z+a)}{r_2^5} - \frac{\mu}{\lambda+\mu}\frac{y}{r_2^3}\right\},$$

$$w_2 = 2\mathrm{A}\left\{\frac{3z(z+a)^2}{r_2^5} + \frac{\mu}{\lambda+\mu}\frac{z}{r_2^3} + \frac{\lambda+2\mu}{\lambda+\mu}\frac{a}{r_2^3}\right\}.$$

Hence the displacements, which are a solution of the problem, may be obtained as follows:—

$$u = u_1 - u_2 = \mathrm{A}\left\{\frac{x}{r_1^3} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{x}{r_2^3} - \frac{6xz(z+a)}{r_2^5}\right\},$$

$$v = v_1 - v_2 = \mathrm{A}\left\{\frac{y}{r_1^3} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{y}{r_2^3} - \frac{6yz(z+a)}{r_2^5}\right\},$$

$$w = w_1 - w_2 = \mathrm{A}\left\{\frac{z-a}{r_1^3} + \frac{\lambda-\mu}{\lambda+\mu}\frac{z}{r_2^3} - \frac{\lambda+3\mu}{\lambda+\mu}\frac{a}{r_2^3} - \frac{6z(z+a)^2}{r_2^5}\right\}.$$

It can be independently verified that these expressions satisfy the equations of equilibrium and the boundary conditions.

One may next investigate the case of the upwardly directed point-push, which will be taken as before to be situated at the point 0, 0, a. In an infinite solid its displacements would be given by

$$u = -\frac{\mathrm{B}x(z-a)}{r_1^3}; \qquad v = -\frac{\mathrm{B}y(z-a)}{r_1^3}$$

$$w = -\mathrm{B}\left\{\frac{(z-a)^2}{r_1^3} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{1}{r_1}\right\}.$$

This stress may be supposed to be balanced by an equal but downward point push, situated at 0, 0, $-a$. We thus obtain

$$u_1 = \mathrm{B}\left\{\frac{-x(z-a)}{r_1^3} + \frac{x(z+a)}{r_2^3}\right\}; \qquad v_1 = \text{etc.};$$

$$w_1 = \mathrm{B}\left\{\frac{-(z-a)^2}{r_1^3} + \frac{(z+a)^2}{r_2^3} - \frac{\lambda+3\mu}{\lambda+\mu}\frac{1}{r_1} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{1}{r_2}\right\}.$$

These displacements as before cause P_{zx} and P_{zy} to vanish along the plane $z=0$, while the pressure p across this plane ($p = -\mathrm{P}_{zz}$) has the value

$$\mathrm{B}\left\{\frac{4\mu^2}{\lambda+\mu}\frac{a}{r^3} + 12\mu\frac{a^3}{r^5}\right\}.$$

Applying the same method as before it is necessary to find what is the potential of a surface density of electricity on $z=0$, equal to the above quantity divided by 4π. Using the theory of electric images it may be shown that where z is positive this potential is equal to that of a charge $\frac{2B\mu(\lambda+2\mu)}{\lambda+\mu}$ together with a dipole of strength $2B\mu a$ in the direction OZ, acting together at the point $0, 0, -a$. Its value is therefore

$$2B\left\{\frac{\mu(\lambda+2\mu)}{\lambda+\mu}\frac{1}{r_2}+\frac{\mu a(z+a)}{r_2^3}\right\}.$$

Using the formulæ (3) and (4) the values of u_2, v_2, and w_2 appropriate to the case of the point-push may now be calculated. We obtain in the first place

$$\Pi=2B\left\{\frac{\mu}{\lambda+\mu}\frac{a}{r_2}-\frac{\lambda+2\mu}{\lambda+\mu}\frac{z}{r_2}-\frac{\mu(\lambda+2\mu)}{(\lambda+\mu)^2}\log(r_2+z+a)-\frac{az(z+a)}{r_2^3}\right\},$$

from which it follows that

$$u_2=2B\left\{-\frac{\mu}{\lambda+\mu}\frac{ax}{r_2^3}+\frac{\lambda+2\mu}{\lambda+\mu}\frac{xz}{r_2^3}-\frac{\mu(\lambda+2\mu)}{(\lambda+\mu)^2}\frac{x}{r_2(r_2+z+a)}+\frac{3axz(z+a)}{r_2^5}\right\},$$

$$v_2=\text{etc},$$

$$w_2=2B\left\{\frac{(\lambda+2\mu)^2}{(\lambda+\mu)^2}\frac{1}{r_2}+\frac{(\lambda+2\mu)z^2+(\lambda+3\mu)az+(\lambda+2\mu)a^2}{(\lambda+\mu)r_2^3}+\frac{3az(z+a)^2}{r_2^5}\right\}.$$

We therefore have

$$u=u_1-u_2=B\left\{\frac{ax-xz}{r_1^3}+\frac{\lambda+3\mu}{\lambda+\mu}\frac{ax-xz}{r_2^3}+\frac{2\mu(\lambda+2\mu)}{(\lambda+\mu)^2}\frac{x}{r_2(r_2+z+a)}-\frac{6axz(z+a)}{r_2^5}\right\},$$

$$v=v_1-v_2=\text{etc.},$$

$$w=w_1-w_2=B\left\{-\frac{\lambda+3\mu}{\lambda+\mu}\frac{1}{r_1}-\frac{\lambda^2+4\lambda\mu+5\mu^2}{(\lambda+\mu)^2}\frac{1}{r_2}-\frac{(a-z)^2}{r_1^3}\right.$$
$$\left.-\frac{(\lambda+3\mu)z^2+4\mu az+(\lambda+3\mu)a^2}{(\lambda+\mu)r_2^3}-\frac{6az(z+a)^2}{r_2^5}\right\}.$$

As before, it may be independently verified that these expressions satisfy the equations of equilibrium and the boundary conditions.

It has next to be shown how these two sets of results may be applied to the cone-sheet and ring-dyke problem. In the first place it may be noted that the displacements, and therefore the stresses they correspond to, have radial symmetry about a vertical axis. It follows that the stresses may be represented as regards direction by the construction of isostatic surfaces, which are everywhere perpendicular to the principal directions of pressure and tension. (The most general types of stress do not of course possess this property.) The term "isostatic surface" is in the present connection a rather unfortunate one. It has nothing to do with isostasy as understood in geology.

One set of isostatic surfaces will be in each case vertical and radial to the axis. The other two systems may be determined by fixing their intersections with the plane of x and z. We have, in fact, $\tan 2\theta = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \Big/ \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)$, where θ is $\tan^{-1}\frac{dz}{dx}$. This equation leads to two sets of curves which are orthogonal The pressures or tensions across the corresponding surfaces are given by the formula

$$P_{nn} = \frac{\lambda u}{x} + (\lambda+\mu)\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \pm \mu\sqrt{\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)^2}$$

which applies only to the plane XOZ. To obtain the pressure or tension across a surface along whose intersection with this plane $\frac{dz}{dx}$ is positive, that value of the root must be selected whose sign is contrary to that of $\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$. The other value refers to the surface which is at right angles to the first, and also to XOZ. Across the plane XOZ itself the tension is given by

$$P_{\phi\phi} = (\lambda + 2\mu)\frac{u}{x} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right).$$

These formulæ, and those for the displacements themselves, may be simplified by assuming $\lambda = \mu$. This corresponds to a value of Poisson's ratio σ which is accurately $\frac{1}{4}$, and the assumption is on the whole justified both by experimental evidence and the facts of seismology. In drawing up figs. 6 to 9, and the table of principal pressures which is given below, this simplification has been used.

Figs. 6, 7 show the directions of the displacements for the point-dilatation system and the point-push system respectively. The amounts of displacement are shown approximately by the lengths of the arrows. Figs. 8, 9 show the corresponding networks formed by the isostatic surfaces. The plane of the diagrams is in all cases XOZ.

3. APPLICATION TO CONE-SHEETS.

The application of these results to the problem of central intrusions is now fairly simple, and I shall deal first with the case of cone-sheets. The broken lines in figs. 8, 9 show isostatic surfaces across which there is a pressure, if, that is, the constants A and B are taken to be positive. One may select any such surface and suppose that the part of the solid within or below it is replaced by a molten magma If this magma exerts the same pressure on the enclosing or superincumbent rock as the substance replaced, the stresses in the remaining solid will be unaltered.

One may, for instance, select that surface in each of the two cases which is distinguished by a shaded margin. It will be noticed that in fig. 8, which shows the type of solution arising from a point dilatation, one is dealing with a closed nearly spherical cavity It must be supposed

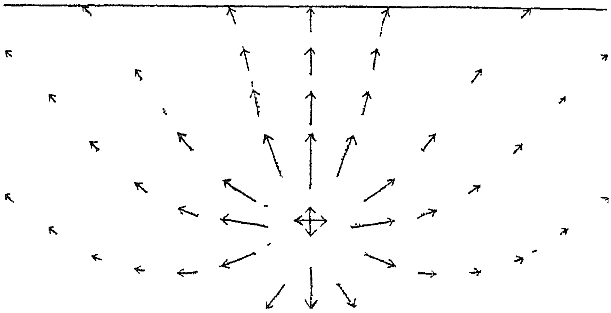

FIG 6 —Point dilatation beneath horizontal surface Dotted lines show direction, arrows proportionate amounts of displacement

that the magma is introduced into this cavity by a pipe of inappreciable width. It must be under such pressure as to cause it to push outwards in all directions. These conditions seem unlikely to be realized in fact, and it is fig 9, giving the solution corresponding to an upward point push, which is put forward as the more probable approximation to reality

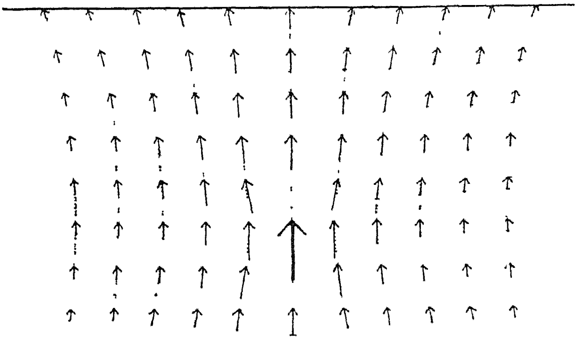

FIG. 7.—Upward point-push beneath horizontal surface. Dotted lines show direction, small arrows proportionate amounts of displacement.

The thin firm lines in figs 8, 9 indicate isostatic surfaces across which there is a tension, and along which one may therefore expect the formation of cone-sheets. In accordance with fact, these increase in steepness towards the centre. Within a certain radius the sheets should also theoretically steepen upwards, and approach the surface in a dyke-like manner (see, however, p. 149). It is unknown whether any of them actually reached it, as the superficial layers have, of course, been removed by erosion. Beyond the radius they should flatten upwards, and become sill-like. This is perhaps the explanation of certain horizontal intrusions,

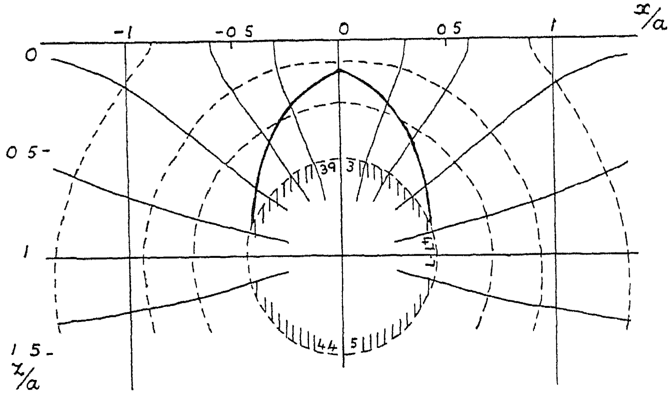

FIG. 8.—Isostatic surfaces produced by point dilatation beneath free horizontal surface Further explanation, see text

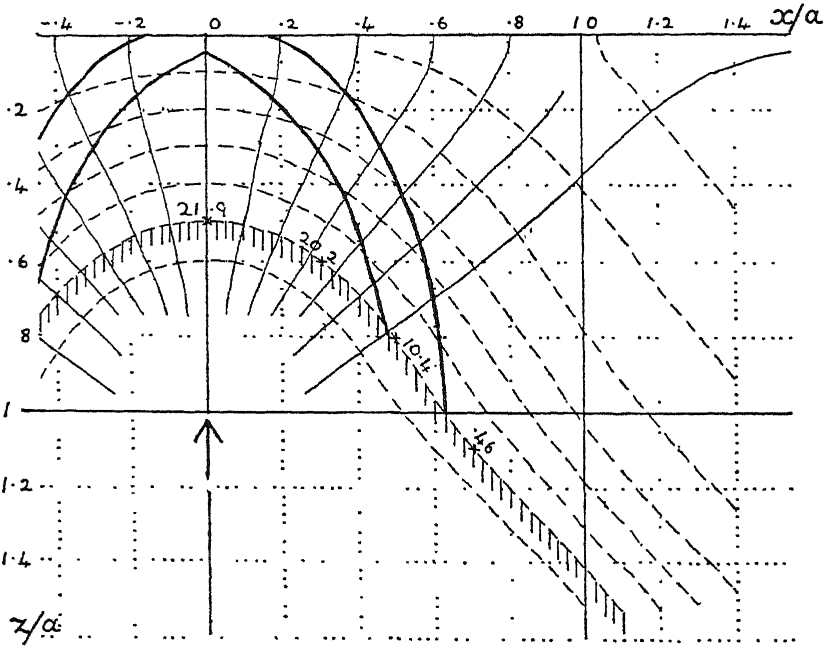

FIG. 9—Isostatic surfaces produced by vertical point-push beneath free horizontal surface. Further explanation, see text.

of typical cone-sheet type, which I have mapped, on the outer fringe of the area invaded from Centre 1 in Mull. The change in behaviour of the outer members was anticipated in the diagram which was published in the Mull memoir (fig. 5).

Figs. 8, 9 also show proportionally at certain points the amounts of magma pressure which are implied in the two solutions. In fig. 9 these diminish rapidly in depth, and farther down it is evident that they would decrease asymptotically to zero. This is important as showing that the shape of the caldron, beyond a certain point, is probably inessential. The same thing might be deduced from the likeness of the two sets of results All that is necessary for a cone-sheet tension seems, in fact, to be an upward pressure on some type of dome.

In speaking of the pressures and tensions which result from the formulæ, it has so far been assumed that there is a weightless medium. In reality, however, the medium is not weightless, and the matter is further complicated by the possibility that there may have been pre-existing lateral pressures or tensions. The simplest possible supposition is that which was made in the Mull memoir, namely, that the density of the intruded rock is uniform, and that before the period of intrusion there was everywhere a "hydrostatic" pressure, equivalent to the superincumbent weight This extra pressure has no effect on the directions of the isostatic surfaces, as it leads to no alteration in the values of $\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$ or $\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}$. It has, however, to be added to the pressures which are indicated in the magma basins, and combined with the stresses corresponding to the chosen value of A or B. It will be instructive to enquire whether the approximate amount of B can be calculated for a particular case, assuming these simple conditions, and the validity of the solution which corresponds to the point-push.

The value arrived at will depend altogether on the nature of another assumption. It is sometimes supposed that dyke formation, for instance, can only occur under actual tension. This need in any case only be small, as the tensile strength of a rock is always much less than its compressive strength. It might be taken as from 30 to 50 bars, which are the limits given for granite in the *International Critical Tables* (vol. ii, New York, 1927) This is, however, not the only possible hypothesis, as when molten fluid is intruded into a crack under pressure it must exert a wedging action. It is possible that tension only exists across the edges of such cracks, and that in this way dykes may propagate themselves perpendicularly to what are not actually directions of tension, but only of minimum pressure. The necessary condition might be that the magma itself was at a pressure a certain amount greater than this minimum. This has to be taken account of in estimating the constant, and it will be found that as a matter of fact the cone-sheet theory will itself afford a criterion which will help to decide between the two hypotheses.

To evaluate B one may use in the first place a datum given by Richey (1932 *b*, p. 129), that the cone-sheets round Centre 2 in Ardnamurchan would converge, if they continued downwards with their surface inclination, at a depth of about 3 miles. The fine firm lines in fig. 9 radiate roughly from a point a little below 0, 0, a, in the lower part of the diagram. The latter point may therefore at present be less than 3 miles in depth, but allowing for denudation a may well be taken as approximately 3 miles or 5 km. (5×10^5 cm.). The subterranean reservoirs were situated well below the surface basaltic lavas, and probably in some type of crystalline metamorphic rock. The density will be assumed to be 2·7, which is slightly above the specific gravity of granite.

It will now be of advantage to compare fig. 9 with the table below, which gives the principal pressures and tensions due to an upward point-push at a given depth, at different points in the surrounding rock. These are equivalent to the stresses caused by the upward and outward pressure of the assumed magma. They refer, however, to a weightless medium, and do not take into account any "hydrostatic" pressure. The figures shown are to be multiplied by $\frac{B\mu}{a^2}$, μ being taken to be equal to λ. a is in the present case 5×10^5 cm., and B the constant to be evaluated.

From an inspection of fig. 9 it will be seen that, if cone-sheets are to be formed, as they have been in the Ardnamurchan instance, at angles of less than 45°, the process of intrusion must have spread outward to at least a certain distance from the central axis of the diagram. The present ground-level may be at a depth of from 1 to 2 km. below the original surface, that is from $\cdot 2a$ to $\cdot 4a$. On the level $\cdot 5a$ intrusion must have taken place at least as far as $\cdot 7a$ from the axis. From the table on p. 147 it will be seen that at the point $\cdot 7a$, $\cdot 5a$ the factor by which $\frac{B\mu}{a^2}$ has to be multiplied to obtain the cone-sheet tension is 4·37

We shall now make the assumption, which may or may not be justified, that this tension was large enough at least to overcome the "hydrostatic" pressure, which existed independently of the system of intrusion. At 2·5 km. in depth this pressure would be about $6{\cdot}6 \times 10^8$ dyne/cm.^{2}, or 660 bars. We thus get $4{\cdot}37 \frac{B\mu}{a^2} = 6{\cdot}6 \times 10^8$, or $B\mu = 3{\cdot}78 \times 10^{19}$ dynes. The upward force of the point-push which gives the displacements which have been assumed is expressed by $12\pi B\mu$ or $1{\cdot}42 \times 10^{21}$ dynes. This is equal to the weight of 1450 cu. km. of water, or 480 cu. km. of basalt, and is only slightly greater than the total superimposed upward force, exerted by the magma on the walls of the caldron.

TABLE OF PRINCIPAL TENSIONS AND PRESSURES.

The figures are only accurate to within one or two units in second decimal place

	0	·1a	·3a	·5a	·7a	·9a → x
0	12·50 12·50 0·00	12·07 12·27 0·00	9·13 10·62 0·00	5·32 8·28 0·00	2·24 6·08 0·00	0·38 4·37 0·00
·1a	8·99 8·99 − 0·92	8·81 8·92 − 0·93	7·30 7·86 − 0·94	4·78 6·27 − 1·00	2·93 4·84 − 0·75	1·43 3·60 − 0·66
·3a	5·94 5·94 − 7·08	6·00 5·84 − 6·94	6·06 5·16 − 5·89	5·34 4·20 − 4·26	4·04 3·29 − 2·78	2·74 2·54 − 1·76
·5a	6·14 6·14 − 21·93	6·20 5·92 − 20·66	6·16 4·55 − 13·86	5·48 3·27 − 7·70	4·37 2·36 − 4·16	3·24 1·81 − 2·30
·7a	12·38 12·38 − 72·78	11·30 10·74 − 62·02	7·66 5·14 − 25·43	5·64 2·64 − 9·60	4·33 1·68 − 4·26	2·84 1·26 − 2·58
·9a			9·60 3·92 − 23·97	5·80 1·50 − 6·14	4·41 1·00 − 2·34	3·36 0·76 − 1·19
1·1a ↓ z			22·40 − 2·14 − 4·34	9·40 0·18 − 1·14	5·34 0·50 − 0·47	3·82 0·57 − 0·19

Figures when multiplied by $\frac{B\mu}{a^2}$ denote principal tensions and pressures caused under free horizontal surface ($z=0$) by point-push of amount $12\pi B\mu$ at point 0, 0, a, upward if B is positive. A positive product denotes tension, a negative pressure. First figure in each square refers to traction across direction of cone-sheets (see text and fig. 9), second across plane of table ($y=0$), third across surface perpendicular to two former.

B itself has the dimensions of an area; if μ is assumed to be $2{\cdot}7 \times 10^{11}$ dyne/cm.^{2} B is about $1{\cdot}4 \times 10^8$ cm.^{2}. The best way to gain some idea of the magnitude of the forces involved is, however, as follows. The pressure at the summit of the caldron shown in fig. 9 would be $21{\cdot}93\frac{B\mu}{a^2}$ or about 3300 bars, equal to that of a column of 34 km. of water. One has therefore to consider the mechanism by which such a pressure may have been produced.

The most obvious cause of upward pressure is a defect in density of the magma, in comparison to the surrounding rock. Liquid basalt at the temperature of fusion was found by Day, Sosman, and Hostetter to have a density of about 2·59 (1914), and if the magma, through a process of differentiation, was more acid than basalt, it might be considerably lighter. One supposes the liquid to have been at the "hydrostatic" pressure corresponding to its depth at some point much farther down in its reservoir than the levels with which we are dealing. If then it was less dense than its walls, the pressure would decrease less quickly upwards than it would in the main body of the rock. There would thus come to be an excess of pressure in the magma, increasing towards the surface The effect is analogous to that which is noticed in the distribution of coal gas, which, as is well known, has a higher effective pressure in the upper-levels of a town. In the case of cone-sheets, however, there is a limit to the excess of pressure, imposed by the possible variations in density, and the size of the caldrons Even if the density of the liquid were as low as 2·4, to produce 3300 bars the caldron would need to have been 112 km. in depth

Such a depth is extremely improbable, and the upward excess of pressure was certainly only a small fraction of the amount which has been deduced. The same must apply to the cone-sheet tension The discrepancy would hardly be removed, even if a difference in shape of the caldron caused the relative intensity of the latter to be increased. The assumption on which the figures have been calculated is therefore probably untrue It is unlikely that any actual tension can have preceded the formation of the cone-sheets.

This result is perhaps of some importance in the general mechanics of intrusion, but it need not have been unexpected While dykes may conceivably in some cases be produced under tension, the same cannot apply to horizontal sills. The latter must, in general, split apart the strata, in spite of a vertical pressure corresponding to the weight of the overlying rock.

It appears, therefore, that there was no actual tension, but only a diminution of pressure, across the directions which have been followed by the cone-sheets. These took the path of least resistance; that is, they opened fissures along planes across which the transverse pressure was least. In the circumstances the value of B becomes difficult to estimate. The upward pressure in the caldron must still have been considerable, though much less than that which was calculated at first.

A further consequence of the particular solution corresponding to the point-push remains to be discussed. If the cone-sheet tensions on any horizontal level are examined in the table they are seen always to be large, and sometimes to reach their maximum, where $x=0$; that is, directly above the centre of the caldron. This does not, however, correspond with reality. The sheets of any particular system occupy a belt one or two miles in width. The central area is often obscured by later ring-dykes, which make observation difficult, but the sheets are never present there so far as can be judged. This suggests that the actual outline of the caldron is a little different from that which corresponds to the point-push. If the real shape were something intermediate between that of fig. 9 and an inverted flower-pot, the concentration of the cone-sheets could perhaps be explained. Under increased magma pressure the sharp edge might cause a local intensification of the tension. This would die out rapidly upwards, but might nevertheless effect the localization of the intrusions in the manner which has been observed.

The second figure which is given for each point in the table refers to the tangential tension, which, if it had been greater than the cone-sheet tension, would have led to the formation of radial dykes. In this respect the solution which has been arrived at corresponds to what, in most cases, is the reality, as the cone-sheet tension is almost everywhere the greater. It has to be noted, however, that the rule is reversed at the surface and at the level $\cdot 1a$, which in the example chosen would have been at a depth of half a kilometre. The rocks to beneath this level have undoubtedly been removed by erosion, so that it is impossible to say whether this is a real effect or due only to an imperfect solution of the problem A slight change in the shape of the caldron might in any case have caused a more general reversal, and in some such manner the Spanish Peaks example of radial dyke formation may be explained.

The different inclination of the cone-sheets of different systems was referred to in the introductory section, but has still to be discussed Thus the sheets of Centre 2 in Mull are, partly at least, steeper than those which surround Centre 1. Round Centre 2 in Ardnamurchan Richey gives the average angle of the earlier sheets as from 35° to 40°, and that of the later sheets as from 65° to 70° This may perhaps be explained as follows. In treating the two solutions it has so far been assumed that a "hydrostatic" pressure only is superimposed. If in addition there is a relative regional tension, say in all horizontal directions, the results will be somewhat modified. The directions of principal pressure in the plane XOZ are given by the formula, which has been already quoted

$$\tan 2\theta = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \Big/ \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right).$$

The tension would leave unaltered the values of such differentials as $\frac{\partial u}{\partial z}$ and $\frac{\partial w}{\partial x}$, but it would increase $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ and diminish $\frac{\partial w}{\partial z}$ The numerator of the fraction would thus be unaltered; the denominator would either be increased or given a smaller negative value. It follows that the surfaces of pressure perpendicular to the cone-sheets would be flattened, while the cone-sheets themselves would be steepened. This would affect somewhat the shapes of the underground caldrons which correspond to the two simple solutions If one adhered to the old shapes the stresses would become more difficult to calculate, but at a distance from the caldrons they would certainly be altered in the way which has been indicated.

It is thus suggested that the latest cone-sheet formation, both in Mull and Ardnamurchan, took place under a condition of regional tension, which was less marked, or perhaps absent, during an earlier period. The chronology of the two cases may be the same. The fact that an epoch of ring-dyke formation intervened in both districts between the earlier and later sheets may also possibly have some significance.

The earlier sheets in Mull form a horse-shoe which, if completed, would be roughly circular. The area invaded by the later sheets is, on the other hand, somewhat elongated. It measures about 8 miles by 6, and the longest direction is from south-east to north-west. This suggests a relative regional tension which was greatest in amount from south-west to north-east. This explanation has already been put forward by Richey, and such a tension is known to have occurred both before and after the entire period of surface vulcanicity and central intrusion, giving rise at both epochs to north-western dykes These dykes form the Mull and Skye "swarms" among others, and extend for long distances to north-west and south-east.

4. APPLICATION TO RING-DYKES, STOCKS, AND SUBSIDENCES

Every text-book on the strength of materials recognizes the existence of two distinct types of fracture. There is, in the first place, shearing fracture, oblique to the direction of maximum tension, or least pressure, and forming angles with this direction of from 45° to 70°. There is, secondly, what may be termed tensile fracture, at right angles to a tension. A mainly tensile stress does not always, however, give rise to this variety, and so far extremely little is known about the circumstances which cause sometimes the one type and sometimes the other to be produced.

It may be regarded as certain that both types occur in the earth's crust, and have to be taken account of in geology. Faults are certainly shear fractures, while ordinary dykes are always assumed to be tensile, on account of their direction, which, except in the case of subsequent tilting, is seldom very far from the vertical. There can be no doubt that this assumption is right, though subject to the proviso which has been mentioned in the last section. There are, in fact, cases where inclined normal faults and vertical dykes have almost certainly arisen under the same horizontal tension (Anderson, 1905)

The subject is important from the present standpoint, as in explaining central intrusions it is necessary to invoke both types of rupture, the tensile type to produce cone-sheets, and the shearing type to account for the production of ring-dykes, by first giving rise to nearly vertical ring-fractures If there had been tangential movement along a cone-sheet fracture, it seems necessarily to follow that this must have been an upward and not a downward movement of its hanging wall—that is, of the side which is next the centre. Otherwise it is obvious that there would have been "jamming," especially if one considers the great number of these fractures This fact rules out one set, and limits consideration to the steeper of the two sets of possible shearing planes which might have resulted from the pressures and tensions considered in the last section. But if these steeper shearing planes had been selected by the cone-sheets, the latter could not have been so little inclined as they sometimes are. This is true at least with regard to the point-push solution. It is therefore fairly certain that the cone-sheets followed the actual planes across which there was minimum transverse pressure.

In the case of ring-dykes the matter is different. To explain these at all one must suppose, as has been already mentioned, that there was a defect instead of an excess of pressure in the magma caldron. This can be mathematically expressed by supposing that A or B is negative. The centre of dilatation is replaced by a centre of contraction, or the upward point-push by a downward one. The fine firm lines of figs. 8, 9 then denote surfaces of pressure, while the broken lines correspond to surfaces across which there is a relative tension. Their directions will not be altered, and with change of sign the table of principal pressures can be applied to the case of the downward point-push. We may select the same shaded boundaries for the magma caldrons, and the defects of pressure will vary along these like the excesses of pressure.

It will be seen at once that the broken lines are in general too little inclined to represent ring-dykes. These must therefore, if they are to be explained at all, have been produced in the first place by shear fracture. In rocks as a whole such fractures are inclined at an average of about 65° to the directions of tension or minimum pressure. This is illustrated by normal faults, produced when the minimum pressure was horizontal, and usually dipping at about this angle. There are thus in the present instance two possible sets of shearing planes at each point, one of which is everywhere inclined about 25° more than the broken lines, and in an outward direction. This is obviously the set which we should select. The thick firm lines of figs. 8, 9 have been drawn at the appropriate angles, and in their lower parts they may well form a representation of ring-dykes.

These lines have an inward curvature, flattening considerably upwards, and such a flattening has not been observed, at least with regard to ring-dykes. The inclination of these is, however, somewhat difficult to measure. An individual dyke can seldom be followed as much as one kilometre in height, and the parts actually seen were originally a fair distance below the surface of the ground. What may be an example of such a deflection is shown by the boundary of the Castle Peak granite stock (Daly, 1912, fig. 40) This is at least suggested by the curve which Daly has figured.

The first stage in the production of a ring-dyke was always, one may suppose, the formation of an arcuate fault, with downthrow towards the centre. The further process need not, however, in every case have been identical. If the arc was incomplete the downthrow must have been a relatively small one, and the segments may have been separated at first only by a fault breccia Into this fault breccia the intruding magma may have forced its way, without causing much further subsidence Such a sequence of events is suggested by the behaviour of the apophysis given off from the Glen More ring-dyke. It is unlikely, however, that any of the wider dykes can be explained in this manner. In these cases it seems necessary to suppose that, even if the dyke itself does not form a closed curve, the fault must have done so. The included block must also have been free to sink It may have extended to the surface. It may alternatively have tapered to a point, or have been separated from the rocks above by the formation of a cross-fracture. If the original fault had an outward inclination a gap would be formed, on one or all sides, during the subsidence, and it is by the filling of such gaps, it is suggested, that most ring-dykes have been produced.

In cases like that of the dykes surrounding the Beinn Chaisgidle centre, in Mull, a separate fault may have been responsible for each ring-dyke. These intrusions here sometimes adjoin, but often they are separated by "screens" of the intruded rock. With regard to some of the Ardnamurchan ring-dykes, however, such a conclusion is improbable. Screens are here almost absent. It seems, in fact, as if the filling of a fissure had not prevented it from remaining as a plane of weakness. After each of the dykes had formed, there was subsidence again along its inner margin, leading to the production of a further ring-dyke. In this way the radius of the intrusions would gradually become less, and the inner dykes would be the later, in accordance with the field evidence. A single fault may thus have been responsible for a whole series of ring-dykes. One may express this differently, and say that these were successively formed by the downward movement of a subsiding block.

It is uncertain, however, whether the broader of the Ardnamurchan ring-dykes can be explained in this manner. If they can there must have been very large downward movements of the subsiding block, or an abrupt tapering These intrusions may, on the other hand, have been originally parts of flat sheets which were intruded along cross-fractures. Their inner margins may be later ring fractures, which gave rise in turn to further intrusion. This is at least a third manner in which ring-like masses may have been produced.

The Beinn a' Ghràig granophyre in Mull may indicate a fourth method. This slightly arcuate intrusion has mainly vertical sides, and seems to have been arched over by an irregular roof. It has been suggested that in this case there was subsidence between two ring fractures. Both faults would according to theory be formed with an inward downthrow, but the direction of motion of the inner one may afterwards have been reversed.

Ring-dykes do not extend so far from their centres in a horizontal direction as do cone-sheets. In the case of Centre 2 in Ardnamurchan, which has already been quoted, cone-sheets are found up to 7 km. from the centre, while ring-dykes cease within 4 km. In the table of principal pressures given on p. 147, the first figures may now be taken to refer to the pressures which are transverse to the original cone-sheet directions, and the last figures to the superimposed tensions. Both these have to be combined with the "hydrostatic" pressure, and, as in the case of cone-sheet formation, it is certain that there can be no absolute tensions. Fractures are most readily formed where a certain quantity with a linear relation to the greatest and least pressure attains a maximum If the two pressures be denoted by P and R, this quantity is not, however, simply the difference $P - R$ It has the form $P - nR$, where n is a constant which may be as large as 5·8 (Anderson, 1934, p. 6)

This quantity will be influenced by the "hydrostatic" pressure which depends on the depth, but for any particular level the position of its maximum may be calculated from the superimposed pressures and tensions which are recorded in the table. The maximum will be seen to occur at each level where $x = 0$, or directly above the centre of the caldron. This, again, does not correspond to reality, as in the Ardnamurchan example, for instance, a fracture must have been formed at one depth about 4 km. away. To overcome this difficulty one may suggest, as before, that the outline of the caldron had something of the nature of a projecting angle. The maximum superimposed pressures might then have been localized above this, as one supposes the tensions to have been during the formation of cone-sheets.

For the shape of caldron shown by the shaded margin in fig. 9, and the corresponding magmatic pressures, the maximum tendency to fracture will occur at the point 0, $\cdot 5a$. Assuming as before that $a=5$ km., the total lateral pressure is here 660 bars $+ 6\cdot 14 \frac{B\mu}{a^2}$. The superimposed vertical tension cannot, however, exceed the "hydrostatic" pressure, as the magma cannot exert an actual suction upon the rock; in other words, its pressure cannot fall below zero. One has therefore $21\cdot 93 \frac{B\mu}{a^2} \ngtr 660$ bars, and with this as a limitation one finds that the horizontal pressure cannot exceed 845 bars. Even if there is no transverse pressure, however, fracture will not take place unless the main pressure exceeds the ordinary crushing strength of the rock, which for granite is about 800 bars Two conclusions may apparently be drawn from this calculation. The magma in the basin must have been at a pressure much below the "hydrostatic" pressure for any particular level, and at the top this may have fallen nearly to zero. There is also an additional reason for supposing that the shape of the caldron is somewhat different from what has been represented. The point-push solution shows how some of the conditions of central intrusion can be represented mathematically, but it can only be a first approximation to reality.

The steeply dipping outer margins of granite stocks may in general have been initiated as ring-fractures, but the present theory does not give an explanation of these, which can be regarded as complete. It might be objected that to explain a stock in terms of the point-push solution one had to assume a previously existing basin which was itself more or less of a stock. The origin of the basins will be discussed in the last section, but the point is not fully met. A difficulty of another type is involved in the explanation of surface subsidences. The ring-fracture directions flatten upwards, and if such a fracture were initiated close to the surface its angle of dip would be only about 25°. The fault which bounds the Glencoe subsidence was, however, originally vertical. It may have originated at some depth and gradually extended upwards. In doing so it may have retained its original direction, but it must be admitted that the dynamics of the process has not been worked out.

5. ORIGIN OF THE MAGMA RESERVOIRS.

The theory of central intrusion which has been discussed explains some of the facts, and something may be learnt from what it fails to explain. In particular it seems that no uninterrupted dome could have been acted on by a magma so as to explain the distribution of either class of intrusions. The reservoirs must have had somewhat the shape of an "inverted flower-pot"

The existence of some type of basins has so far been taken for granted, but no theory will be complete which does not account for their origin At present, however, this is mainly a matter of conjecture One may imagine the original reservoir to have had the form of a flat sheet, of no great thickness, but at a considerable depth If the method of intrusion of ordinary dykes is vertical, such a sheet must have extended in early Tertiary times from the Hebrides at least as far as Yorkshire

The magma must then have worked its way up into the crust in certain areas in more or less stock-shaped forms. This could not probably have been by selective melting. This hypothesis would require a large degree of horizontal inhomogeneity in the crust, and inhomogeneities seem not to be in the horizontal direction so much as the vertical. It would appear rather that at certain spots there must have been selective stoping, or subsidence of blocks into the liquid layer on a larger scale.

It is difficult in most cases to see why, in what must have been large areas, only a few particular spots were selected. But the liquid layer could not have been formed without irregularities, and some of these would probably be local, though maybe only gentle projections into its roof. Suppose now that after these were formed there was a period of cooling, with partial solidification of the magma. The projecting areas might become more or less separated as lenticular pockets, and the roof of the layer might every here and there be in contact with its floor. It is easy to see what would then have happened if there had been further consolidation. The diminution of volume which this entailed would cause a very great reduction in the pressure of the magma. This might originally have corresponded roughly to the weight of the overlying rock. If it became small enough the roofs of the pockets would be bound to sink, and possibly at least they might sink by fracture

In this way the upward penetration of the liquid layer into the crust may have begun. One may suppose that there were many periods of consolidation, each followed by remelting, and each of them may have carried the process a little further. A similar partial solidification may have caused the loss of pressure in the magma basins which was referred to in the last section. If the basins still communicated, the influence may have extended from one to another, and so, as has been suggested, the ring-dykes of the Beinn Chaisgidle centre in Mull may be contemporaneous with the earlier of those which surround Centre 2 in Ardnamurchan.

These suggestions are, of course, purely speculative, and the theory of central intrusion is still far from complete An attempt has only been made to sketch the outlines of this very complex problem.

I am indebted to the Director of the Geological Survey and the Controller of the Stationery Office for permission to reproduce figs. 1, 3, and 5; and in the case of fig 2, to the above and to the Council of the Geological Society of Glasgow, as well as to Dr Richey. With regard to fig. 4, the Council of the Geological Society of London and Dr Richey have kindly given similar consent

6. SUMMARY.

It has been shown how the displacements due to a point dilatation are modified, when the point is in the neighbourhood of a free plane surface. The same has been done in one case for the strain corresponding to the "premier type simple d'intégrales," here renamed the point-push. The conclusions arrived at might possibly have other applications, besides those in geology.

The geological application is as follows. An upward point-push at a certain depth might have caused the relative tensions which have given rise to a system of cone-sheets It is unnecessary, however, for the thrust to have been concentrated at a single point. The same solution will apply to the upward and outward pressure exerted by a magma in a certain shape of basin. This, as a first approximation, probably represents the facts, though the actual contour of the basins must have been a little different from those which are deduced from the formulæ.

A downward point-push, at a corresponding depth, might have caused shear fractures to develop in the directions now followed by any system of ring-dykes, and this might also have been effected by a falling off of pressure in the supposed basin The greater width of ring-dykes may be explained, if the fractures were closed in cross-section The blocks which they surrounded might then have become completely detached, and they widened downwards. If they sank into the magma, they would leave large vacant spaces along their sides, to be filled by the intrusions.

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(*Issued separately August* 7, 1936.)

X.—Note on Fracture. By Dr Harold Jeffreys, F.R.S.

Communicated by the GENERAL SECRETARY.

(MS. received July 3, 1936. Read July 6, 1936.)

DR ANDERSON'S paper (*Proc. Roy. Soc. Edin.*, vol. lvi, 1936, pp. 128–157) is of considerable interest as a contribution both to geology and to the theory of elasticity. One of his assumptions about fracture, however, has little support, and I am inclined to think that some difficulties that he mentions towards the end would be lessened if it were removed. This assumption is that the complete form of a fracture surface can be inferred directly from the distribution of stress before fracture, intersecting the surfaces of principal stress at some constant angle. It is often to be found in both geological and mechanical literature; but it rests on no theoretical grounds and is, I think, untrue.

The usual Coulomb-Hopkins theory of fracture (*Camb. Phil. Trans.*, vol. viii, 1849, pp. 456–466) asserts that fracture begins at the time and place where the difference between the greatest and least of the principal stresses first reaches the strength of the material. On the more recent theory of Mieses the stress-difference is not the only determinant; but the function whose value determines whether fracture will occur or not is so nearly proportional to the square of the stress-difference that very special experimental conditions are needed to find out which criterion is the better. Taylor and Quinney have shown that ductility in copper agrees best with the Mieses criterion (*Phil. Trans.*, A, vol. ccxxx, 1932, pp. 323–362); but Taylor has shown further that in certain conditions the two criteria will actually give the same result.

In Dr Anderson's problem the strength is first reached around a circle in contact with the magma reservoir, and nowhere else. In practice minor irregularities will cause it to be reached first at only one point. If fracture ever spreads beyond one point, therefore, it is evident that the stresses must have changed since the first fracture began; and therefore the previous stress is no guide to the spreading of the fracture unless the extra stresses happen to be proportional to the original ones. There is no evident reason why they should be; on the contrary, the first fracture introduces a new boundary, near which the stresses are drastically altered at once, while they are little changed at a distance from it. The form of the fracture therefore depends both on the original stress and on the changes introduced as the fracture develops As fractures can extend into regions where the original stress-difference was only a small fraction of the strength, the second part must be the larger in such regions

The actual process of fracture is still imperfectly understood. A sudden change of stress would be expected to send out elastic waves of P and S (longitudinal and transverse) types, but apparently these cannot be the means of extending the fracture further. The stresses in them must fall off inversely as the distance travelled, and as the original stress-difference was only equal to the strength the waves cannot carry stresses enough to break the material anywhere else. But the local stress-differences are increased It must be remembered that when a crack occurs the general stress is still there, and any lack of uniformity increases the stress-differences in equilibrium. Even near a spherical hole the greatest stress-difference is about twice as great as it is several radii away. With an elongated crack the effect at the ends is much greater. Thus the sending out of elastic waves does not restore equilibrium. The general stress is as before and the local stresses near the ends of a crack are much greater. But in this region the general stress-difference was almost up to the strength, and the equilibrium stresses in presence of the crack would much exceed the strength. Hence the fracture must extend, presumably with a velocity less than that of an S wave, though this is not proved, and it appears that it is only in special conditions (where the original stresses are wholly internal) that it will fail to reach the outer surface.

Elongated cracks probably play an important part in determining the original fracture. It is found by experiment that perfect crystals yield only by continuous distortion; fracture depends on imperfections of structure. Near a spherical hole in a solid under non-hydrostatic stress, again, the greatest stress-differences are on the principal planes, and such a hole would therefore extend along one of these planes This would be satisfactory for tension cracks, but not for fractures inclined to the principal stresses. G. I. Taylor has, however, shown that an elongated crack will account for both types (*Proc. Roy. Soc.*, A, vol. cxlv, 1934, pp. 1–18). He treats the problem as two-dimensional, there being no displacement along the intermediate principal stress. If the other dimensions are given, and the crack is empty, it produces the greatest stress-differences if the longer is across the greatest tension. If it is filled with a fluid with the same compressibility as the solid, the greatest stress-difference occurs if the longer dimension is at 45° to the greatest principal stress. In both cases the greatest stress-difference is at the edges of the crack, where the distortion is greatest, and the crack will therefore extend in its own plane. The results are the same if the Mieses criterion is used instead of the stress-difference. The cracks most easily extended on this criterion have their longest axes in the direction of the intermediate principal stress. It is easily seen that the extension or compression due to a stress along the crack is little different close to the crack from what it is anywhere else; thus even if the Mieses criterion is correct on a molecular scale it still leads to the old stress-difference criterion for a body as a whole.

The fluid filling a crack must be regarded as a model. It is recognised that local irregularities of structure exist, such as interfaces between separate crystals. The atoms near these have several possible positions of equilibrium and can attach themselves to either crystal, according to the system of stress. Such a structure has been shown to give a qualitative explanation of elastic afterworking (Jeffreys, *Proc. Roy. Soc.*, A, vol. cxxxviii, 1932, pp. 283–297). The disordered atoms will resist a general compression as much as ordered ones, but their readiness to switch over into new orientations can be imitated roughly by assuming zero rigidity—that is, fluidity. If the stress as a whole is compressive the tendency is to close the crack, and is resisted by the quasi-fluid; but if it is a tension the fluid can become detached or split, and the behaviour will then be as for an empty crack. Irregularities therefore provide an explanation of the difference in direction between tensile and compressive fractures.

In an earlier paper (*Trans. Geol. Soc. Edin*, vol. viii, 1905, p. 387) Dr Anderson has investigated the consequences of supposing that the criterion for fracture is that $P - \mu N$ shall reach S; where S is a property of the material analogous to the strength, μ another property analogous to a coefficient of friction, P the shear stress and N the normal stress (supposed compressive) across a possible plane of fracture. The criterion on the Coulomb-Hopkins theory is simply $P \geqslant \frac{1}{2}S$. Anderson's generalisation has much observational support, but since Taylor's work may need reinterpretation Interfaces between crystals exist before the application of stress, and run in all directions. Taylor's theory identifies the directions of the cracks most likely to extend; but these are always at 45° on the theory as far as he has taken it. The actual angle to the greatest (compressive) stress is greater than 45° and is explained by Anderson's theory. But Taylor assumes that the critical value of the Mieses function is independent of the mean stress, which is correct for some materials The work of Bridgman, however, has shown that strength usually increases with the mean compressive stress. Anderson's result is not inconsistent with Taylor's. He is considering the conditions in fracture when it becomes complete; Taylor is considering them when it begins; and there is a gap to be filled in.

It might appear that if the critical value of the Mieses function for rocks depends on the mean stress Taylor's result would be modified, but this is not the case. The mean stress with a full crack gives no departure from hydrostatic pressure, and the direction of the first crack that will extend will be unaltered. The amount of the general stress-difference needed would of course depend on the mean stress, but the directions would remain as before.

The theory, however, is only for a single plane crack. In an actual solid the cracks curve about in all directions, and a crack at 45° will soon have to change its direction. But the longer it becomes the larger is the Mieses function at its ends corresponding to a given general stress-difference. Thus when displacement has occurred along one crystal face it will be able to continue along another or along a cleavage plane in a somewhat different direction, and there is a possibility that the active cracks may unite. The solid is at this stage honeycombed with active cracks, and there is no reason why they should fail to extend right across it. Then part of it will move together in such a way as to relieve the general stress, which up to this stage has remained unaltered. It is only at this stage that there is a definite fracture surface, and the conditions are completely altered. The statical theory no longer applies, as the possible internal stresses are limited to the strength and are insufficient to balance the external forces that produced the original general stress, and the motion will be accelerated. It appears reasonable, however, that the direction will be that of easiest motion. For a given displacement the work done by the external forces will be proportional to the shearing stress, but a portion proportional to the normal stress is used up in overcoming friction. Thus the work available to be turned into kinetic energy will be proportional to $P - \mu N$, in Anderson's sense. This will be greatest for displacements in the direction that he finds. There appears, however, to be no question of this exceeding a critical value, since the only critical value that can arise in the problem is certain to be exceeded at this stage.

Since the final fracture is a matter of mass motion it will be resisted least if it is a possible motion of a pair of solids in contact. Now let us imagine sliding in a bar crushed by stresses at the ends if the fracture surface was curved. As soon as a little motion has taken place the ends would be separated in some places and tightly rammed together in others. In the latter the local stresses would be increased and the matter squeezed into the hollows; thus the curvature in the direction of slip would tend to be removed. The fracture as finally developed would therefore tend to be plane, or at least a ruled surface, even if the original surface was curved. This appears to explain why fracture surfaces are usually as smooth as is consistent with the micro-structure of the material.

A complete theory would evidently be a very complicated matter, and at present we cannot even write down the necessary equations We can, however, say that there is no inconsistency between the Mieses-Taylor theory and Anderson's, which refer to different stages in fracture, and that the form of the fracture surfaces is determined by changes of stress arising during fracture and not by the original distribution of stress

Anderson finds (*Proc. Roy Soc. Edin* , vol. lvi, 1936, p 152) that the lines of fracture should have "an inward curvature, flattening considerably upwards, and such a flattening has not been observed." From the above considerations, however, there appears to be no reason why it should. The curvature is found from the surfaces of principal stress before fracture; the actual fractures would probably be much more nearly cylindrical or even inclined outwards

It may be remarked that the conditions of a point-push are closely imitated when a pointed tool is placed in contact with a block of flint and struck with a hammer. What happens is that a complete cone falls out, though the statical stresses must be very like those given by Anderson

Seismology has hitherto paid little attention to what actually happens at the focus of an earthquake, being concerned mainly with the waves after they have left it. Some facts, however, have come to light that will need to be taken into account in a fuller theory of fracture In some earthquakes the S waves arrive at distances up to 600 km. or so a little earlier than they should had they started at the same time as the corresponding P waves. I have suggested as an explanation that in these earthquakes the original fracture sent out only S movement, P being generated only by reflexion of S at the outer surface and therefore being somewhat delayed. Objection has been made to this explanation; on the other hand no alternative explanation of the fact has been offered The discrepancy is a matter of 3 seconds, and if the observations are as accurate as they appear to be its uncertainty cannot be put at more than 1 second (*M.N.R.A.S. Geophys. Suppl.*, vol. iii, 1933, p. 142). It is not common to all earthquakes; in some deep-focus ones in particular the P and S movements can be shown to have left the focus within a second of each other.

Deep-focus earthquakes themselves offer a difficulty in relation to the extension of Anderson's theory. They occur at depths up to about

700 km., 300–400 km. being fairly common At this depth the pressure is about 100 times the strength of rocks at the surface, while for solid friction μ is usually a moderate fraction, say 0·6. Thus apparently the shearing stress at these depths must be of the order of 100 times the strength of surface rocks if it is to overcome friction We know of no cause for such stresses. Bridgman actually finds that the stress-difference needed to give permanent distortion increases with pressure, but most of the materials he has tested so far give increases of 2 to 20 per cent. of the rise of pressure (*Phys Rev*, vol. xlviii, 1935, pp. 825–847), up to 50,000 atmospheres, corresponding to a depth of about 150 km in the earth. The fractures in these earthquakes do not, of course, reach the surface, the stresses that produce them being presumably accommodated locally; but the motion produced is as strong as in normal earthquakes, and suggests that at these high pressures the coefficient of friction must decrease considerably Anderson's term μN is clearly needed at moderate pressures, as friction has to be overcome in any sliding movement, but apparently μ must diminish considerably at high pressures and temperatures.

(*Issued separately August* 7, 1936.)

XI. — The Relation of the Pituitary Gland to Muscle Creatine. By B. G. Shapiro and H. Zwarenstein. *Communicated by* Professor LANCELOT HOGBEN, F.R S.

(MS received January 18, 1936. Revised MS. received July 6, 1936
Read July 6, 1936)

BRAIER (1931) showed that hypophysectomised dogs on a complete diet excrete 24 per cent. less urinary creatinine than the controls, and 35 to 40 per cent. less when fasting. Schrire and Zwarenstein (1933, 1934) found that injection of anterior lobe extracts increased the urinary creatinine in rabbits.

These results afford evidence of an endocrine relationship between the anterior lobe of the pituitary and urinary creatinine. In view of the well-established relationship between urinary creatinine and muscle creatine the present authors decided to investigate the relationship of the anterior lobe to muscle creatine. An effect on phosphocreatine of muscle was reported by Marenzi (1933), who obtained a decrease of 33 per cent following removal of the anterior lobe of the pituitary. The effect of hypophysectomy and of injections of anterior lobe extracts on muscle creatine has not been previously investigated.

The South African Clawed Toad, *Xenopus lævis*, was chosen as the experimental animal because of its ability to withstand severe operative procedures. While removal of the pituitary gland in mammals is an operation of great difficulty and is associated with a high mortality rate, in *Xenopus lævis* it is performed with ease, the mortality is low and the animals survive the operation for a year or more Moreover, the characteristic colour response of the animal serves as a useful indication of the completeness of removal of either one or both lobes of the pituitary (Hogben and Slome, 1931).

The research was conducted in two stages:

1. Removal of (*a*) anterior lobe of the pituitary; (*b*) both lobes of the pituitary.
2. Injection of anterior lobe extracts.

EXPERIMENTAL.

Several hundred animals were collected and kept in large open air-tanks in the laboratory.

Muscle creatine was estimated in animals fresh from the pond in order to determine (*a*) the normal concentration in males and in females, and (*b*) to ascertain if any difference exists between the creatine content of the muscles of the right and left hind limbs. These estimations were necessitated by the fact that the creatine content of the muscles of *Xenopus lævis* had not been previously investigated.

During the same period as the above determinations were being made the anterior lobe of the pituitary was removed from about 50 animals. Half of these animals were males and half were females A similar batch was totally hypophysectomised. A batch of unoperated animals was kept under the same conditions as the operated animals and served as captivity controls. In addition the operative procedure itself was controlled by drilling a small hole into the roof of the mouth near the site of operation for the actual removal of the pituitary lobes.

The muscle creatine was estimated (*a*) 6–10 weeks, and (*b*) 18–22 weeks after operation.

The extract of anterior lobe used was prepared according to the method of Bellerby (1933). Oviposition is induced in the South African Clawed Toad by the preparation, and this was used as an indication that an active extract had been injected. Control extracts of brain tissue were prepared in the same way.

In all cases the hamstring muscles were used and the creatine content estimated according to the method of Rose, Helmer, and Chanutin (1927).

RESULTS AND DISCUSSION.

All figures for muscle creatine are expressed as milligrams total creatinine per 100 g. fresh tissue Total creatinine includes preformed creatinine, which, however, does not exceed a few milligrams in muscle.

(a) *Normal Muscle Creatine.*—

TABLE I.—MUSCLE CREATINE IN NORMAL POND ANIMALS.

Males.		Females.	
Right Leg.	Left Leg.	Right Leg.	Left Leg.
402 ± 24	403 ± 22	396 ± 21	396 ± 22

Each figure represents the mean and the standard deviation of 16 determinations.

It is seen that there is no difference between the values for right and left legs. There is no significant difference between the muscle creatine of males and females.

The average, inclusive of right and left legs of both sexes, is 399 ± 21 mg. per 100 g. muscle. This figure, based on 64 determinations, represents the average normal creatine content of the hamstring muscles of *Xenopus lævis*. Taking the average of the right and the left leg of any individual toad as the true value for that animal, the muscle creatine of the toads was found to range between 349 and 440 mg per 100 g. About 80 per cent. of the toads gave values ranging between 375 and 425 mg. per 100 g.

(b) *Hypophysectomy*.—A preliminary estimation on toads that had been hypophysectomised 6 months previously showed that there was a decrease of about 15 per cent. in muscle creatine as compared with captive controls. The results obtained were as follows: Captive controls, 368 mg.; anterior lobe removed, 311 mg.; both lobes removed, 317 mg. In a further series of experiments muscle creatine was estimated 6–10 weeks and 18–22 weeks after operation. The creatine content of the hamstring muscles of the right and the left legs in each animal was always separately estimated, the result for one leg serving as a check on the accuracy of the result obtained for the other.

TABLE II —MUSCLE CREATINE AFTER HYPOPHYSECTOMY.

	Captivity Control	Operative Control.	Anterior Lobe Removed	Both Lobes Removed.
6–10 weeks after operation .	399	401	375	376
18–22 „ „ .	380	..	333	313

Each figure represents the mean of the values obtained for the right and the left legs in 10 animals, *i.e.* the mean of 20 determinations. As in the case of normal toads, there was no difference between the right and left legs or between males and females.

6 to 10 weeks after hypophysectomy there is a decrease in muscle creatine, but the difference is probably not significant. Statistical analysis of the figures for muscle creatine 18–22 weeks after the operation shows that both the figures for partial and for total hypophysectomy are significantly lower than the figures for the controls, but are not significantly different from each other The average decrease after hypophysectomy is 15 per cent.

(c) *Injection of Anterior Lobe Extract*—Determinations were made in a series of normal animals and the results coincided with the figures obtained in the previous year (Table I), the average of right and left legs being 395 mg. per 100 g. muscle, and the range 352 to 436 mg.

The creatine content of the muscles of (*a*) normal animals, (*b*) animals injected with anterior lobe extract, and (*c*) animals injected with brain extract, was estimated Only those animals which oviposited after injection with anterior lobe extract were used for muscle creatine estimations. First, the effect of a single intraperitoneal injection of a large dose (1 ml.) of extract was investigated. 1 ml. of brain extract was injected as a control 12 animals were injected in each case 2 animals from each set were killed off at intervals of from 20 minutes to 48 hours after injection and the muscle creatine determined The muscles of the right and left legs were used for estimation in each animal Neither brain nor anterior lobe extract had any significant effect on muscle creatine. The effect of repeated injections of smaller doses of extract was therefore investigated. Animals were injected daily with 0·2 ml. each of anterior lobe extract, and the muscle creatine determined at intervals from 1 day to 3 weeks after injection. A parallel series of brain extract injections (0·2 ml. each daily) served as controls. Brain extract had no effect. 8 days after injection with anterior lobe extract one toad gave a value of 463 mg , which is definitely above the upper limit of the normal range. 14 days after injection the muscle creatine content was 468 mg., and 21 days after the commencement of injection it was 506 mg., 31 per cent. above the average value for the brain controls (390 mg.).

The second experiment was repeated, injection being continued up to 6 weeks on this occasion. In this case a maximum increase of 29 per cent. was obtained 32 days after the commencement of injections. Animals hypophysectomised 11 months previously were given daily injections of anterior pituitary extract. This led to an increase in muscle creatine, the values approaching those for normal animals which had been kept in captivity for 11 months The number of animals was, however, too small to provide definite evidence.

The results obtained after hypophysectomy and after injection of anterior lobe extract suggest that an endocrine relationship exists between the anterior lobe of the pituitary and muscle creatine. The long latent period before the effects of hypophysectomy and injection appear indicates that the relationship is probably an indirect one through some other endocrine organ. It is interesting to note that although in one set of experiments toads were injected with anterior pituitary extract for 3 weeks and in another set of experiments for 6 weeks, no additional response was obtained. Collip and Anderson (1935) found that continued injection of anterior pituitary extract containing the thyrotropic hormone resulted in the formation of a refractory period within 2–3 weeks.

Since removal of both lobes had no greater effect than removal of the anterior lobe alone, it is probable that the effect of hypophysectomy is due solely to the removal of the anterior lobe.

SUMMARY.

1. In the normal South African Clawed Toad (*Xenopus lævis*) the creatine content of the hamstring muscles is the same for the right and the left legs. There is no difference in the muscle creatine level in the two sexes. The average value is 399 mg. creatine per 100 g. muscle. The highest value obtained for a normal toad was 440 mg., and the lowest 349 mg.

2. There is a decrease of 15 per cent. in muscle creatine 18–22 weeks after hypophysectomy. The decrease after total hypophysectomy is not significantly different from that after removal of the anterior lobe alone.

3. Injections of 0·2 ml. anterior lobe extract daily results in an increase of about 30 per cent 3–5 weeks after the commencement of injections. Up to this time no refractoriness was observed.

4. The evidence presented is suggestive of an endocrine relationship between the anterior lobe of the pituitary and muscle creatine. This relation is probably an indirect one through some other endocrine organ.

The expenses of this investigation were defrayed by a grant from the Research Grant Board of the Union of South Africa, to whom the authors express their thanks.

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(*Issued separately September* 1, 1936).

XII.—The Brackish-water Lochs of North Uist.

By **Edith A. T. Nicol,** B.A., Ph D., The Department of Natural History, The University of St Andrews. *Communicated by* THE PRESIDENT.

(With Seven Figures.)

(MS received May 19, 1936. Read July 6, 1936)

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INTRODUCTION

ABOUT the year 1695 Martin Martin visited the Hebrides, and of the Island of North Uist he writes, "There is such a number of fresh water lakes here as can hardly be believed. . . . They are generally well stocked with trout and eels and some of them with salmon, and, which is yet more strange, cod, ling, mackerel, etc , are taken in these lakes into which they are brought by the spring tides." This old reference suggested that the brackish-water fauna of the Hebrides might be of considerable interest and extent. Consequently I spent part of the summers of 1933 and 1935 in North Uist in order to study the fauna of the lochs and the conditions under which the animals were living.

I wish to record my thanks to Professor A. D. Hobson for his interest and help in the work. I am also indebted to Mr R. Elmhirst of Millport, Dr C. H. O'Donoghue and Miss de Wattville of Edinburgh University, to members of the Entomological Department of the Natural History Museum, South Kensington, and to Mr A. R. Waterston of the Royal Scottish Museum for help in identifying the animals; also to Miss M. D. Dunn of St Andrews University for identifying the algæ. The meteorological records were obtained from the Air Ministry.

Literature.

M'Intosh (1866) visited North Uist and gives a short description of the fauna of the "inland seas," listing about a dozen species of animals found in them.

Harvie-Brown and Buckley (1888) give an account of the vertebrates of North Uist, including the fishes

Murray and Pullar (1910) during the survey of the Scottish fresh-water lochs made an extensive series of soundings in several brackish lochs in the Hebrides and also give short accounts of the flora and fauna.

Stephen (1935) describes the distribution of the burrowing animals in the intertidal zone in North Uist.

Methods.

All the lochs were visited more than once and stations chosen to represent as wide a range of salinity and as many types of bottom as possible. Owing to the lack of boats only the shore and shallow water fauna could be investigated, but plankton hauls were made from the shore and algæ collected at each station. Wherever possible, water samples were taken at a depth of 6 in. and at the bottom, and the hydrogen ion concentration and temperature also recorded. Differences in composition of the water at each station were insignificant except at the mouths of streams, and in the tables only the average value is given. Salinity was estimated with silver nitrate, alkali reserve with N./100 HCl using methyl orange as indicator and pH colorimetrically.

THE ISLAND.

North Uist lies in the middle of the chain of islands known as the Outer Hebrides and is separated from Dunvegan Head in Skye by the 16 miles of sea which form the Minch. The main island is about 13 miles long by about 18 wide; but has many outlying islands, the chief of which are Boreray and Vallay in the north, Kirkibost and Baleshare on the west, and Grimsay on the south. On the north it is separated by the Sound of Harris from the Island of Harris, and on the south from the Island of Benbecula by the North Ford, which at low water forms a maze of sand and rocky islets over a mile in width, which can be crossed on foot.

The island is composed entirely of gneiss overlaid with peat, often to a depth of 15 ft., but on the low-lying north and west coasts stretches of "machair land" are formed by the mixing of blown sand and comminuted shells with the peat, so that instead of water-logged bog and sparse heather, a coastal strip of fertile grazing and arable land is formed, thick with clover, ladies' bedstraw and many other flowers. The rest of the island is in sharp contrast. On the east, and to a certain extent on the north, there are hills, Eaval reaching a height of 1138 ft. The centre lies low, often only a few feet above sea-level, barren and desolate, consisting of such a tangle of bogs and lochs of irregular outline and shallow depth that there appears to be more water than land. These lochs drain for the most part into Loch Eport and Loch Maddy, long intricate arms of the Minch which penetrate to within a few hundred yards of the Atlantic on the western and northern sea-boards.

THE RAINFALL.

The rainfall of North Uist is relatively low, averaging about 45 in per year The driest months are April, May and June. The following table gives the rainfall for the years 1932–35.

TABLE I.

Year.	Jan.–March	April–June	July–Sept	Oct –Dec.
1932	10 48 in.	4·81 in.	17·59 in	14 97 in.
1933	12·27 ,,	7 60 ,,	11·30 ,,	9·60 ,,
1934	10 96 ,,	8·13 ,,	16·10 ,,	13 65 ,,
1935	11·33 ,,	5 73 ,,	11·52 ,,	15·34 ,,

THE CONDITIONS IN THE LOCHS.

In certain parts of the island peat containing the roots and branches of forest trees underlies the sand of the seashore, and stretches down at least as far as low-water mark (Beveridge, 1926). This clearly indicates that the land has sunk, and may still be sinking, so that many of the lochs which once were fresh are being invaded by the sea. These lochs number 18 altogether, varying in area from about 300 sq. yd. (Oban Alioter) up to 1·17 sq. miles (Loch Obisary). Most of them are only a few feet in depth, but in some there are deep holes. Murray and Pullar (1910) sounded many of them, and give full details in their *Bathymetrical Survey*. Narrow channels connect the lochs with the sea, and in several places they have been rendered still more narrow by dry stone embankments built to carry the roads.

The Salinity.

The salinity of the lochs during the months of June and July 1933 and 1935 was as follows :—

TABLE II.

Loch.	Height above Sea-level.	Average Salinity °/oo. 1933. Springs.	Average Salinity °/oo. 1933. Neaps	Average Salinity °/oo. 1935. Springs.	Average Salinity °/oo. 1935. Neaps
Loch Leodasay	6·0 ft.	.	2·4	7·7	2·8
Loch Obisary	8·2 ,,		10·4	.	10·5
Loch Strumore	3·7 ·	.	..	..	24·0
Loch nan Geirran	7·5 ,,		..	.	4·0
Loch an Duin:					
Outer Basin	..	37·9	..	..	34·2
Middle Basin	4·5 ,,	27·0	.	..	25·4
Inner Basin	..	4·4	.	..	5·4
Oban Alioter	.	.	.	.	28·5
Oban a' Chlachain	3·75 ,,	28·2	27·9	31·0	29·4
Oban na Curra	.	.	16·8	..	..
Oban Irpeig	..	..	..	34·1	33·2
Oban nam Fiadh:					
Basin I		.	24·4	34·0	25·2
,, II	..		21·6	33·6	20·3
,, III	.	.	17·8	21·2	17·6
,, IV	.		16·4	18·0	16·1
,, V	5·2 ,,	.	11·6	10·5	10·2
Oban Trumisgarry I	..	..	.	26·2	..
,, ,, II	..	.	.	32·0	.
Loch an Sticer	.	34·9	.	..	20·9

The Alkali Reserve and pH.

The alkali reserve of the lochs varies between 0·0027 N. and 0·0004 N., the salter lochs having the higher value. All the drainage water from the land percolates through peat and has consequently a low alkali reserve of only about 0·0002 N., this being the average value for a number of streams. The high value of the alkali reserve of the brackish water must, therefore, be due to the sea water entering the lochs. This conclusion is supported by the figures given in Table III, in which a comparison is made between the observed values of the alkali reserve in waters of different salinities, and the calculated values based on the assumption that this factor is controlled by the ratio between sea water of 0·0026 N. (average observed value) and fresh water of 0·0002 N. (average observed value).

TABLE III.

Salinity °/oo.	Per cent. Sea Water.	Observed Alkali Reserve.	Calculated Alkali Reserve
35·5	100·0	0·0026 N.	.
30·2	85·7	0·0022 N.	0·0023 N
25·0	70·4	0·0018 N.	0·0019 N.
16·0	45·0	0·0011 N	0·0013 N.
10·0	28·0	0·0009 N	0·0009 N.
5·2	14·6	0·0006 N.	0·0006 N.
3·0	8·4	0·0004 N	0·0004 N.

A small amount of calcium carbonate may be derived from molluscan shells, and, in certain lochs on the machair land, from blown shell sand. The effect of this is shown in Loch an Sticer where at a salinity of 30·6°/₀₀ the alkali reserve was as high as 0·0027 N.

The hydrogen ion concentration of the lochs shows considerable variations between pH 7·8 and 9·9. The pH of the fresh-water lochs averages about 5·4. The brackish-water lochs with the lowest salinity have also the lowest pH owing to the large admixture of acid water draining through the peat. The salter lochs have a much richer vegetation and their higher pH is probably due to this factor as well as to the higher salinity (Nicol, 1935).

The Temperature of the Water.

No detailed observations were made on the temperature of the water, but during the summer temperatures as high as 28° C are by no means uncommon among weed. The average temperature of the lochs during July 1935 was 20° C. During the winter, on the other hand, the lochs may be covered with a sheet of ice for considerable periods. This occurred during the winter of 1932–33 and again in 1935–36.

The Nature of the Bottom.

The bottom of the lochs varies considerably in consistency. In parts of Loch an Sticer and Oban Irpeig, for instance, the bottom is formed of shell sand mixed with a small proportion of comminuted gneiss. In Oban Trumisgarry, Loch Obisary, parts of Loch an Duin, etc., the bottom sand is formed only from gneiss; in others there may be a deposit of peat detritus, or else a fine black mud mostly derived from peat and decaying seaweed.

The water of the lochs tends to bring into solution all calcareous matter present in the bottom, such as blown shell sand and local dead shells. The process is begun even in living molluscs whose shells are greatly eroded in the lochs but not on the strands.

THE ECOLOGY OF THE LOCHS.

One of the most interesting parts of North Uist is the Clachan district on the west coast, containing the sand and channels of Clachan Strand, Oban Irpeig, Loch Leodasay and Oban a' Chlachain (fig. 1).

The Strand.

The strand consists of sand fringed by a narrow belt of saltings and traversed at low tide by channels carrying the overflow water of the two

Obans and of the fresh-water loch, Loch nan Struban. The sand is fine and consists of fragments of shells mixed here and there with powdered gneiss and traces of peat soil. At low tide the strand dries out, except for small pools and channels, for 7 or 8 hours at a stretch. At high-water neaps it is covered to a depth of several inches except at the margins and over a central sandbank; at high-water springs the sand is all covered to a depth of 18 in. or 2 ft.

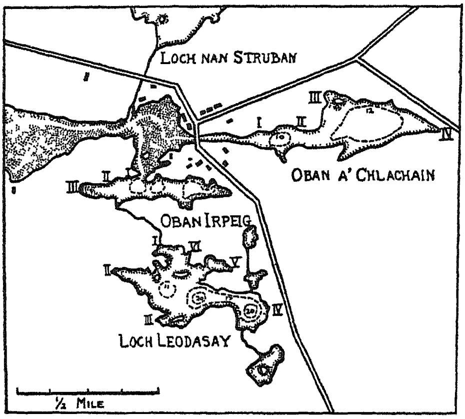

FIG 1 —Map of the Clachan district to show the lochs and the position of the sampling stations.

The sand-burrowing fauna of this region has been investigated by Stephen (1935) and consists of *Macoma balthica* (L), *Cardium edule* L. and *Arenicola marina* L In addition *Corophium volutator* (Pallas) occurs in patches along the edges of the channels

On the surface of the sand the gastropod *Peringia ulvæ* Pennant is abundant and also the small nudibranch *Alderia modesta* (Lovén), which was breeding in early August in 1933. In pools on the surface of the sand *Crangon vulgaris* L., *Neomysis vulgaris* (J V. Thompson) and *Gobius microps* Kröyer are abundant, while in the channels *Praunus flexuosus* (Müller) largely replaces Neomysis. *Pleuronectes flesus* L and *P. platessa* L. up to 8 or 10 in. long, and the young of *Gadus virens* L., *Anguilla vulgaris* Turton and *Gasterosteus aculeatus* (L.) are also common in the channels.

Round the margins where stones occur *Fucus vesiculosus* L. and *Ascophyllum nodosum* Le Jol grow luxuriantly and harbour a fauna of *Littorina littorea* (L.), *L. littoralis* (L) and *L. saxatilis* var. *rudis* (Maton) Under the stones *Nereis diversicolor* O. F. Müller occurs along with small *Carcinus mænas* (Pennant), *Lipura maritima* Guérin and *Gammarus marinus* Leach, while in the sandy banks of the saltings *Enchytræus albidus* Henle and *Paragnathia formica* (Hesse) are present.

The presence of Alderia, Gobius and Peringia indicates a brackish water condition of the strand. After a summer thunderstorm the salinity of the small surface pools may be as low as $8\text{–}10^{\circ}/_{00}$, while the effect of the incoming tide is to dam back the overflow water from the Obans, so that except in periods of abnormal drought the salinity of the water at high tide will be somewhat below that of the open sea.

Oban Irpeig.

Oban Irpeig is a small loch about half-a-mile long, which drains out at low water on to the south side of the strand by a channel about 20 ft. wide and 30 yd. long, crossed by stepping stones, and situated in the middle of the north side of the loch. Opposite, the Leodasay burn flows into the Oban. The greater part of the loch is shallow, not more than 8 or 10 in. deep at low-water neaps, but with considerable variations in depth at different stages of the tidal cycles. At three points, however, the water reaches a depth of at least 8 ft., although accurate soundings have not been made At the two ends the bottom is of soft black mud rich in decaying organic matter, and becomes gradually sandier towards the middle. Although the margins there will support one's weight fairly well, it is not possible to cross from one side to the other.

The level of this loch is more affected by the tides than that of any other, as the north side near the entrance is low and at the top of the spring tides water enters by a number of wide gaps. Table IV gives the depth of water at three stations (fig. 1) during neap and spring tides.

TABLE IV

	Station I		Station II.		Station III	
	Low Water.	High Water.	Low Water.	High Water.	Low Water	High Water.
Neap tides . .	18 in	24 in	9 in	13 in	3 in.	9 in.
Spring tides .	12 ,,	55 ,,	2 ,,	42 ,,	dry	30 ,,

Table V shows the salinity, pH and alkali reserve of the water at the same stations during spring tides on 25th June and neap tides on 1st July 1935.

TABLE V.

	Station I.			Station II.			Station III.		
	Sal. ‰	pH	Alkali Reserve.	Sal ‰.	pH.	Alkali Reserve.	Sal. ‰.	pH.	Alkali Reserve.
Spring tides	34·4	8·2	0·0026 N	33·8	8·2	0·0024 N.	32·2	8·2	0·0023 N.
Neap tides .	33·4	9·0	0·0025 N	32·6	9·2	0·0024 N.	32·2	9·2	0·0023 N.

Oban Irpeig is edged with stones or with bare rock faces on which grow *Pelvetia canaliculata* Decne. and Thur. and *Fucus vesiculosus* along with an occasional clump of stunted *Ascophyllum nodosum.* Well-grown Ascophyllum is abundant near the outflow, and in the deeper water there *Fucus serratus* L., *Chorda filum* Lamour and *Laminaria saccharina*

Lamour are common along with a certain amount of *Halydris siliquosa* Lyngb., *Cladophora Balliana* Harv. and *Ceramium echionotum* J. G. Agardh. In all the shallow-water area, except the muddiest parts, the bottom is covered with a thick growth of *Ruppia maritima* L., *Zostera marina* L. and in parts *Z. nana* Roth. The mud is covered with a short dense growth of *Vaucheria litorea* Bang and Agardh., and floating masses of green weed occur on the surface of the water or are thrown up at high-water mark. These consist largely of *Percursaria percursa* Rosenv , *Chætomorpha linum* Kütz. and *Cladophora glaucescens* Harv. *Fucus ceranoides* L., *Ulva lactuca* var. *latissima* DC. and *Chætomorpha litorea* Cooke occur at the mouth of the Leodasay burn.

The fauna of the loch is rich. At high-water mark under stones and among weed the following occur:—

Ligia oceanica (L.).
Porcellio scaber Latreille
Orchestia gammarella (Pallas).
Talitrus saltator (Montagu).

Salda littoralis L.
Pterostichus nigrita F.
Anthomyid larvæ.

Bdella littoralis L.

The following occur in the open water:—

Gadus virens L.
Salmo trutta L.
Gasterosteus aculeatus (L.).
Atherina sp. (young).
Mugil sp.

Praunus flexuosus.

The mullet are large and very numerous, and although the loch is fished sporadically for sillock and trout, the mullet are never taken on the hook. They used at one time to be netted when a boat was available. Two species are recorded from the Hebrides by Harvie-Brown, *Mugil capito* Cuvier and *Mugil chelo* Cuvier, the former from brackish water, the latter from the sea. Owing to the shallowness of the water and the amount of weed it was not possible to make systematic plankton hauls, but there was no sign of *Aurelia aurita* (L.) or of *Sarsia tubulosa* (M. Sars) which were abundant in Oban a' Chlachain.

The following occur among the Ruppia and Zostera:—

Gonothyræa loveni (Allman).

Lineus gesserensis O. F. Müller.

Idotea viridis (Slabber).
Jæra marina (Fabricius).

Chironomus aprilinus Meig

Hydrobia ventrosa (Montagu)
Littorina saxatilis var *rudis.*
Skeneopsis planorbis (Fabricius).
Cingula semicostata (Montagu)

Corophium volutator (Pallas).
Gammarus duebeni Lilljeborg.
Carcinus mænas (Pennant).

Rissoa parva (da Costa).

Bowerbankia imbricata (Adams).

In the clean sand in the central part of the Oban *Arenicola marina*, *Cardium edule* and *Macoma balthica* are abundant, while empty shells of *Akera bullata* Müller and *Retusa alba* (Kanmacher) occur in large numbers; but although a careful search was carried out only a single living Retusa could be found.

The muddier parts contain numerous Corophium, Nereis and Chironomus larvæ. A few specimens of *Peringia ulvæ* occur on the surface, while *Hydrobia ventrosa* is common in the floating weed. At the mouth of the Leodasay burn, *Gammarus duebeni* is abundant in the weed and under stones, along with the Isopod *Sphæroma hookeri* Leach, as many as thirty-eight of the latter being found under one stone.

The fauna at the outflow of the loch is more definitely marine, the following being collected among the Fucus and Laminaria:—

Halisarca dujardini Johnston.
Halichondria panicea (Pallas)
Chalina limbata (Montagu).

Dynamena pumila (L)
Sagartia troglodytes (Price).

Spirorbis borealis Daudin.

Balanus balanoides (L)
Leander squilla (L.).
Eupagurus bernhardus (L.).
Carcinus mænas.

Littorina littorea.

Asterias rubens L.

Flustrella hispida (Fabricius).

Loch Leodasay.

Loch Leodasay lies to the south of Oban Irpeig and drains into it by a small stream about 300 yd. long (fig. 1). The loch has a total area of 0·08 sq. miles and has an average depth of 7·38 ft. The loch is irregular in outline and there are several arms which are partially isolated from the rest of the loch by stony shallows. The north shore is rocky but elsewhere the bottom is sandy or muddy with few stones.

Although the loch lies 2 ft. lower than Loch Obisary it is entered by fewer tides, only the new moon springs reaching it owing to the length and narrowness of the outflow burn and the presence of Oban Irpeig. The average salinity, pH and alkali reserve is low. The following table gives these three factors at various parts of the loch on 2nd July 1933 and 28th June 1935, 21 days and 8 days respectively, after a tide had entered the loch.

TABLE VI.

Station	Salinity °/₀₀ 1933.	Salinity °/₀₀ 1935.	pH. 1933.	pH. 1935.	Alkali Reserve. 1933	Alkali Reserve. 1935.
I.	3·6	8·8	7·0	9·4	0·0005 N	0·0006 N.
II.	3·6	3·0	7·0	8·9	0·0004 N	0·0004 N.
III.	1·8	3·0	7·2	8·7	0·0004 N.	0·0004 N.
IV.	1·8	3·0	7·0	8·3	0·0004 N.	0·0006 N
V.	1·2	2·8	8·8	8·5	0·0007 N	0·0005 N

The flora of the loch is fresh water in character. Indeed the only indication that the water is brackish lies in a few tufts of *Enteromorpha micrococca* Kütz. and *Monostroma undulatum* Wittr. near the outflow, both of which are common in the stream itself. In the deeper water *Potamogeton alpinus* Balb. and *Sparganium natans* L. occur sparingly; in the shallow water *Lobelia Dortmanna* L., *Myriophyllum alterniflorum* DC., *Potamogeton natans* L., *Equisetum limosum* L. and *Chara delicatula* Agardh. grow freely, while every stone is fringed with a slimy growth of Zygonema.

The fauna is brackish in character throughout the greater part of the loch. *Orchestia gammarella* and *Porcellio scaber* occur under stones at the edge, while in the loch itself the following brackish-water animals are found:—

Gammarus duebeni.
Jæra marina.
Neomysis vulgaris.

Hydrobia jenkinsi Smith.

Anguilla vulgaris.
Gasterosteus aculeatus.
Pleuronectes flesus.

Gobius microps and *Sphæroma hookeri* occur in a little patch of sand and stones where the stream leaves the loch. In addition to the brackish-water fauna a considerable number of fresh-water species colonise the loch, but not very successfully:

Nepa cinerea L.
Sigara sahlbergi Fieb.
Enallagma cyathigerum Charp.
Gerris costæ H S.

Plumatella repens L.

Pygosteus pungitius (L.).

In the fresher parts of the loch the following are also found:—

Sigara scotti Fieb.
Sympetrum striolatum Charp.
Ischnura elegans Lind.
Leptocerus aterrimus Steph.
Triænoides bicolor Cl.
Limnophilus sp.
Donacia versicolorea Brahm.
Gyrinus natator L.

Limnæa peregra (Müller).
Hydrobia jenkinsi.

Gasterosteus aculeatus.

Loch Leodasay is the only loch in which the plankton is comparable in richness with that of the fresh-water lochs. *Daphnia longispina* (O. F. Muller), *Diaptomus laticeps* Sars and *Cyclops strenuus abyssorum* Sars occur in enormous numbers in the deeper parts of the loch, and one or two specimens of *Eurycercus lamellatus* (O. F. Muller) were obtained in the fresher region. Loch Leodasay is the most interesting of the lochs as it is almost the only one in which a true mingling of the brackish and fresh-water faunas takes place, species like *Jæra marina* and *Neomysis vulgaris* living side by side with animals usually only associated with fresh water.

Oban a' Chlachain.

This is a small tidal loch lying to the east of Clachan Strand and draining on to it. It runs nearly east and west and is three quarters of a mile long. It consists of a triangular eastern part with an average depth of 14 ft. and a narrow shallow western part terminating in an outflow under the road. No streams drain into the loch.

Owing to the narrow outflow, almost entirely blocked by a stone embankment carrying the road, the level of the water is very slightly affected by the daily tides. At neap tides there is a rise of 6 in. in the western, hardly any in the eastern, part. At spring tides the daily rise is about 8 in. and 3 in. in the two parts respectively, but in the western part the level of low-water springs is nearly 2 ft. above that of low-water neaps, and in the eastern part there is a difference in level of 3 in.

The following table gives the salinities, pH and alkali reserve of the two parts of the loch at spring and neap tides on 24th June and 2nd July 1935 respectively:—

TABLE VII.

	Neap Tides.			Spring Tides.		
Station.	Salinity $^\circ/_{oo}$.	pH.	Alkali Reserve.	Salinity $^\circ/_{oo}$	pH.	Alkali Reserve.
I.	28·8	9·1	0·0020 N.	29·4	8·3	0·0019 N.
II	28·6	9·2	0·0019 N.	30·2	8·5	0·0020 N.
III.	28·0	8·9	0·0019 N.	32·2	8·7	0·0020 N.
IV.	30·2	8·9	0·0021 N.	33·4	8·7	0·0023 N.

The flora of Oban a' Chlachain is similar to that of Oban Irpeig. *Zostera nana* and *Ulva lactuca* do not occur, but *Cladophora rupestris* Kütz., *Spermatochnis paradoxus* Kütz., *Furcellaria fastigiata* Lamour, *Ahnfeltia plicata* Fries. and *Chondrus crispus* Lyngb. are found in this loch.

The fauna is similar to that of Oban Irpeig but more varied.

The following species occur in the Laminaria, mainly in the western part:—

Lucernaria quadricornis O. F. Müller.
Actinia aquina L.

Nicolea zostericola (Oersted).

Amphithœ rubricata (Montagu).

Musculus discors (L.).
Littorina littoralis.
Cadlina lævis (L).

Alcyonidium hirsutum Fleming.
Alcyonidium albidum Alder.
Flustrella hispida Fabricius.
Hippothoa hyalina (Smitt).
Tegella unicornis (Smitt).

Dendrodoa grossularia (van Beneden).

Large clumps of *Clavellina lepadiformis* (O. F Müller) occur on vertical rock faces, while the nudibranchs *Actæonia cocksi* (Alder and Hancock), *Limapontia capitata* (O. F. Müller) and *Alderia modesta* occur among the floating weed. *Hemimysis lamornæ* (Couch) is to be found in the loch, as well as all the species found in Oban Irpeig with the exception of *Peringia ulvæ* and *Sphæroma hookeri* which do not appear to be present *Praunus flexuosus*, *Aurelia aurita* and *Sarsia tubulosa* are the only species in the plankton.

Loch Obisary.

Loch Obisary is situated on the east side of the island, and is the largest brackish-water loch in North Uist, having an area of 1·17 sq miles, and a maximum depth of 151 ft., with an average of about 25 ft. (fig. 2) It opens into Loch Eport near its mouth by a narrow stream about 100 yd. long with a fall of 8·2 ft.; consequently only spring tides enter the loch and in summer it has a salinity of about $10^{\circ}/_{00}$, varying little from place to place. The following table gives the values, in various parts of the loch, of the salinity, pH and alkali reserve on 18th July 1933, and on 12th July 1935, during the period of neap tides:—

TABLE VIII.

	Salinity $^{\circ}/_{00}$.		pH.		Alkali Reserve	
Station.	1933.	1935.	1933.	1935.	1933.	1935
I.	10·0	10·0	8·4	8·8	0·0016 N.	0·0009 N
II.	10·4	10·0	8·4	8·6	0·0010 N	0·0009 N.
III.	9·8	10·0	7·8	8·2	0·0010 N	0·0009 N.
IV.	10·8	10·4	7·9	8·6	0·0011 N	0·0009 N.
V.	11·8	11·2	8·4	8·2	0·0012 N.	0·0010 N.
VI.	12·2	11·0	8·8	8·2	0·0014 N.	0·0010 N.
VII.	11·2	.	8·6	.	0·0020 N.	.
VIII.	10·6	..	8·2	..	0·0014 N	.
IX.	9·8	..	7·8	..	0·0014 N.	.
X.	10·4	.	7·8	..	0·0018 N.	

The bottom is in parts stony and in parts covered with a coarse sand of disintegrating gneiss mixed more or less with peat debris. This sandy bottom is frequent on the east and north sides but rare on the west, where the shore is steeper and rockier. No streams of any size enter the loch, but small trickles of fresh water reduce the salinity, and lower the pH locally to about 5·4.

Plants are scarce. At the south end of the loch isolated patches of *Myriophyllum alterniflorum* and of *Potamogeton pectinatus* occur, the latter most plentifully in sheltered inlets with a peaty bottom as at Station III. Towards the mouth of the loch these plants disappear and at Station V. a short poor growth of *Ruppia maritima* along with *Enteromorpha intestinalis* Link. and *Cladophora trichocoma* Kütz. replaces them. Nearer the mouth *Fucus ceranoides* appears for the first time and at the mouth itself *Fucus vesiculosus*, dwarf Ascophyllum, *Cladophora rupestris* and *Catenella repens* Batt. also grow in considerable quantity.

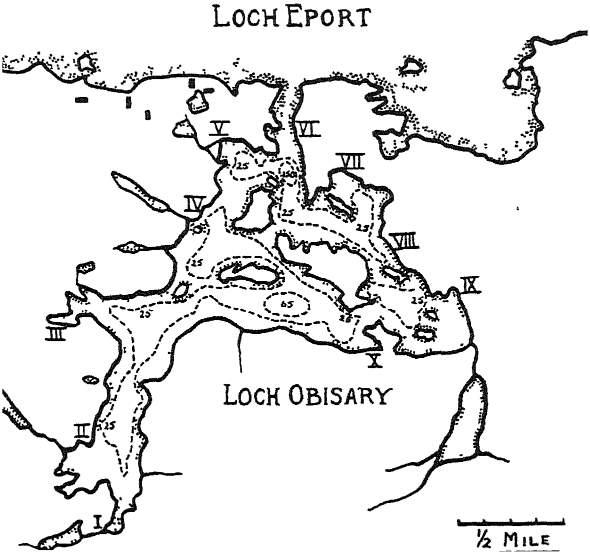

FIG. 2—Map of Loch Obisary to show the position of the sampling stations

The fauna is poor both in species and in numbers. On the north side and at Station I, where the bottom is sandy, the following occur:—

Gasterosteus aculeatus.
Anguilla vulgaris.
Gobius microps.

Corophium volutator.
Gammarus duebeni.

Macoma balthica.
Peringia ulvæ.
Hydrobia ventrosa.

Wherever there are large stones, as on the west shore, *Mytilus edulis* L., *Jæra marina* and *Procerodes ulvæ* (Oersted) are found. *Neomysis vulgaris* occurs at Stations I, II, IX and X, but is replaced nearer the mouth of the loch by *Praunus flexuosus*. *Nereis diversicolor* occurs in almost pure peat at Station V, and *Littorina saxatilis* var. *rudis* at the outflow. Brown trout, sea trout and salmon are fished in the loch. The plankton is uniform throughout, and consists of the copepod *Acartia clausi* (Giesbrecht) in large numbers, and a few medusæ of *Sarsia tubulosa*. Veliger larvæ are abundant; they appear to belong to *Littorina littorea*, which does not occur in the loch, but they could not be identified with certainty.

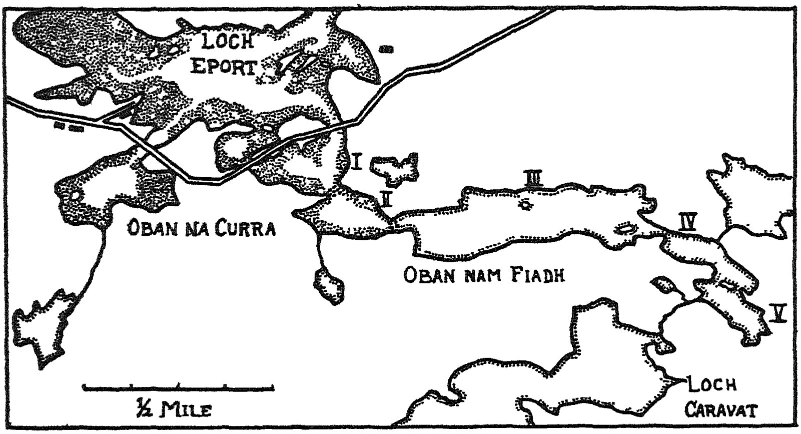

FIG. 3.—Map of Oban nam Fiadh to show the position of the sampling stations.

For some reason at present unknown, neither *Hydrobia jenkinsi*, nor *Littorina saxatilis* var. *tenebrosa* are to be found in the loch. The absence of *Idotea viridis* appears to be correlated with the almost total absence of Ruppia.

Oban nam Fiadh.

This is a narrow arm of Loch Eport, running south-east for a mile and a half from near the head of the loch. The Oban is not all at the same level, but is divided into a number of basins by narrows which must at one time have been the short connecting streams of a series of small lochs (fig. 3). The total area is 0·14 sq. miles. The bottom is of sandy mud, and the shores of the same substance, with stones here and there. The depth is from 4–6 ft. throughout.

The level of the loch, especially in the lower basins, is considerably influenced by the tides. The following table gives the differences of level at spring tides and at neaps:—

TABLE IX.

	Spring Tides.		Neap Tides.	
	Low Water.	High Water.	Low Water.	High Water
Basin I.	15 in.	30 in.	6 in.	12 in.
,, II.	12·5 ,,	23 ,,	8 ,,	9·5 ,,
,, III.	8 ,,	15 ,,	4 ,,	4 ,,
,, IV.	12 ,,	17·5 ,,	8 ,,	8 ,,
,, V.	4·5 ,,	7 ,,	2 ,,	2 ,,

Murray and Pullar imply that the top basin is only 5 ft. 2 in. above sea-level, and should consequently be influenced by every tide; but owing apparently to the elongated and constricted shape of the loch, only spring tides affect the uppermost basin. The same authors state that Loch Caravat also is said to be influenced by the tides, but considering the small effect on Oban nam Fiadh of an ordinary spring tide, and the narrow burn with a rise of over 2 ft. connecting the two lochs, this seems unlikely. The salinity of Loch Caravat is $0{\cdot}2\,{}^{\circ}/_{\circ\circ}$, the same as that of many other lochs removed from all tidal influence, and the fauna shows no signs of periodic flooding with salt water.

Table X gives the conditions in the various basins at neap and spring tides on 27th June and 2nd July 1935.

TABLE X.

	Neap Tides.			Spring Tides.		
	Salinity ${}^{\circ}/_{\circ\circ}$.	pH.	Alkali Reserve.	Salinity ${}^{\circ}/_{\circ\circ}$.	pH.	Alkali Reserve.
I.	25·0	9·7	0·0018 N.	34·0	..	0·0022 N.
II.	20·4	9·5	0·0013 N.	33·0	..	0·0022 N.
III.	17·6	9·9	0·0011 N.	21·6	..	0·0018 N.
IV.	16·0	9·6	0·0010 N.	18·4	..	0·0012 N.
V.	8·8	8·9	0·0007 N.	9·2	..	0·0007 N.

The lowest basin, near the outflow under the road, has stony shores covered with *Pelvetia canaliculata* and *Fucus vesiculosus*. In deeper water Ascophyllum, *Fucus serratus* and *Cladophora rupestris* are abundant. The rest of the basin is shallow with a bottom of muddy sand covered with a thick growth of Ruppia and *Zostera marina*, while towards the edges stunted and deformed *Fucus vesiculosus* var. *balticus* J. G. Agardh. replaces the Zostera in the very shallow water. The following species of animals occur in this basin:—

Arenicola marina.
Nereis diversicolor.

Idotea viridis.

Littorina littoralis.
Littorina saxatilis var. *rudis.*
Littorina saxatilis var. *tenebrosa.*
Peringia ulvæ.

Corophium volutator.
Gammarus duebeni.
Praunus flexuosus.
Carcinus mænas.

Hydrobia ventrosa.

Gasterosteus aculeatus.
Gobius microps.
Anguilla vulgaris.

At the narrows between this basin and the next conditions are different, for stones have been piled up to make a causeway, and water is always flowing in one direction or the other. The vegetation is similar to that at the outflow of the lower basin, and besides the animals already mentioned the following are to be found:—

Gonothyræa loveni.
Clava multicornis (Forskal).

Procerodes ulvæ.

Jæra marina.
Gammarus duebeni.

Mytilus edulis.
Cingula semicostata.
Skeneopsis planorbis.

Bowerbankia imbricata.

Atherina sp (young).

The second basin is similar to the lower one. Between it and the third there is again a causeway. The fauna there is similar to the above with the addition of abundant *Mya arenaria* L., twenty-nine of these molluscs being found in an area of 1 sq. ft. among the stones, and *Hydrobia jenkinsi*, which occurs along with *H. ventrosa* and *Peringia ulvæ.* Clava and Skeneopsis are absent.

The vegetation of the third basin is like the last with the addition of *Potamogeton pectinatus*, but only the following species of animals were observed:—

Gammarus duebeni.
Praunus flexuosus.
Neomysis vulgaris.

Mytilus edulis.
Mya arenaria.
Hydrobia jenkinsi.

Gasterosteus aculeatus.
Gobius microps.
Anguilla vulgaris.

In the lower part of the basin Praunus occurs, and is replaced towards the top by Neomysis.

The fifth basin is again at about the same level but is separated from the last basin by an island and a bed of Phragmites. The Caravat burn flows in on the south side and reduces the surface salinity for a small distance round. The only weeds present are *Myriophyllum alterniflorum* and *Cladophora brownii* Harv., which occurs in enormous quantities. The fauna is much reduced, the following only being found:—

Gammarus duebeni.
Neomysis vulgaris.

Hydrobia jenkinsi.

Gasterosteus aculeatus.
Gobius microps.
Anguilla vulgaris.
Pleuronectes flesus.

Loch Strumore.

This is a large loch on the east side of the island, draining by a narrow outlet under the road into an arm of Loch Maddy. It is over a mile in length, and has an area of about 0·24 sq. miles. Murray and Pullar state that it has a level bottom about 12 ft. deep in the middle (fig. 4).

The loch is entered daily by the tide, and the salinity is uniform throughout, except at the top where a moderately large stream flows in from Loch Fada. Table XI shows the conditions in the loch on 5th July 1935 just after the period of spring tides:

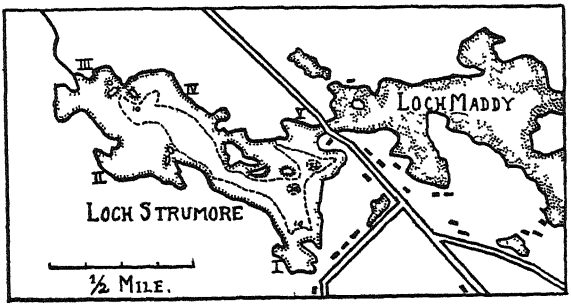

FIG. 4.—Map of Loch Strumore to show the position of the sampling stations.

TABLE XI.

Station.	Salinity °/₀₀.	pH.	Alkali Reserve.
I.	24·8	9·3	0 0017 N.
II.	23 6	9 0	0·0017 N
III.	1·4	8·9	0 0003 N.
IV.	24·2	9 5	0·0015 N.
V.	23·6	9·2	0·0014 N.

The shores of the loch are mostly stony, but there are many shallow, sandy or peaty bays, and considerable stretches of bare rock. All round the margins *Fucus vesiculosus*, *Chadophora rupestris* and *Chondrus crispus* grow abundantly, except at the top, where *F. vesiculosus* is replaced by *F. ceranoides* and Chondrus is absent. In the shallow bays the bottom is covered with a thick tangle of *Ruppia maritima* and *Fucus vesiculosus* var. *balticus*, while Ruppia and *Zostera marina* grow in the deeper water.

The fauna of the loch is uniform throughout, the following species occurring:—

Aurelia aurita.
Sarsia tubulosa.

Idotea viridis.
Jæra marina.
Corophium volutator.
Gammarus duebeni.
Praunus flexuosus.

Mytilus edulis.
Macoma balthica.
Littorina saxatilis var. *tenebrosa.*
Littorina saxatilis var. *rudis.*
Hydrobia ventrosa.
Peringia ulvæ.

Gasterosteus aculeatus.
Anguilla vulgaris.
Salmo trutta.

Hydrobia jenkinsi occurs in the upper half of the loch, a little way up the Loch Fada burn, and on floating weed elsewhere. A single specimen of *Limapontia capitata* was taken in the floating weed.

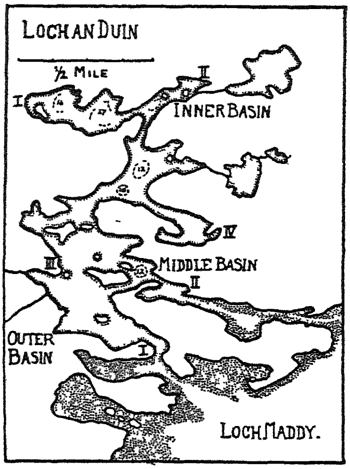

FIG. 5.—Map of Loch an Duin to show the position of the sampling stations.

Loch an Duin.

Loch an Duin is a large loch of irregular outline, draining into Loch Maddy by two narrow outlets, and having an area of 0·19 sq. miles. It is divided into a northern basin of low salinity and a southern basin of high salinity joined by a narrow channel. The southern and less important outlet does not communicate direct with Loch Maddy, but with a small outer basin which drains almost dry during neap tides. The loch is not more than 7 ft. deep except for one or two small areas (fig. 5) and its level appears to be unaffected by the tides except in the small outer basin. The following table shows the conditions in the loch during spring tides on 27th July 1933 and during neap tides on 27th July 1935:—

TABLE XII.

		Salinity °/₀₀.		pH.		Alkali Reserve.	
		1933.	1935.	1933.	1935.	1933.	1935.
Outer basin		34·9	34·2	9·2	9·2	0·0023 N.	0·0022 N.
Middle basin	I.	25·9	25·0	9·2	9·6	0·0019 N.	0·0015 N.
	II.	27·6	26·2	9·2	9·2	0·0016 N.	0·0016 N.
	III.	9·6	10·2	9·2	9·2	0·0008 N.	0·0008 N.
Inner basin	I.	5·2	6·0	8·8	8·8	0·0006 N.	0·0006 N.
	II.	3·0	4·3	8·8	9·1	0·0005 N.	0·0005 N.

In the outer basin the bottom is composed of semi-liquid mud, except at the outlet where it is gravelly, and there are many large stones covered with *Fucus serratus* and stunted *Laminaria saccharina* as well as *Chondrus crispus* and *Ceramium echionotum*. In the muddy part *Akera bullata* is abundant along with larval *Chironomus aprilinus* and *Corophium volutator*; *Peringia ulvæ* and *Hydrobia ventrosa* crawl on the surface.

At the outlet the fauna is very different, and the following species occur:—

Sycon coronatum (Ellis and Solander).
Chalina limbata.
Halichondria panicea.
Halisarca dujardini.

Dynamena pumila.
Tealia felina (L.).
Aurelia aurita polyps

Lineus gesserensis.

Arenicola marina.
Amphitrite johnstoni Malmgren.
Spirorbis borealis.

Balanus balanoides.
Jæra marina.
Idotea viridis.
Gammarus marinus.
Gammarus duebeni.
Amphithœ rubricata.
Praunus flexuosus.

Lipidochitona cinereus (L.).
Musculus discors.
Patella vulgata L
Littorina littorea.
Littorina saxatilis var *rudis*
Littorina saxatilis var. *tenebrosa.*
Hydrobia ventrosa.
Peringia ulvæ.
Cingula semicostata.

Asterias rubens.

Bowerbankia imbricata.
Alcyonidium hirsutum.
Callopora liniata Gray
Schizoporella unicornis Johnston.

Ciona intestinalis L.
Ascidia sordida O F. Müller.
Dendrodoa grossularia.
Clavellina lepadiformis.

Gasterosteus aculeatus.

The southern and eastern shores of the middle basin are rocky and covered with *Fucus vesiculosus* and *Cladophora rupestris* and in a deeper zone *Fucus serratus* and *Chondrus crispus* together with occasional plants of *Laminaria saccharina.* The western shore consists mostly of shallow sandy bays carpeted by a short growth of Ruppia and *Fucus vesiculosus* var. *balticus,* in the deeper parts by Ruppia and *Zostera marina,* with a few patches of *Chorda filum.* In the fresher bays *Fucus ceranoides* and *Enteromorpha intestinalis* occur.

The following animals are found where the shores are rocky:—

Halisarca dujardini.

Procerodes ulvæ.

Mytilus edulis.
Hydrobia ventrosa.
Littorina saxatilis var. *rudis.*
Littorina saxatilis var. *tenebrosa.*

Lineus gesserensis.

Jæra marina.
Gammarus duebeni.

Cingula semicostata.

Bowerbankia imbricata.

In the sandy bays and among the weeds the following occur:—

Arenicola marina.
Nereis diversicolor.
Manayunkia æstuarina (Bourne).

Idotea viridis.
Corophium volutator.
Gammarus duebeni.
Praunus flexuosus.

Macoma balthica.
Mya arenaria.
Cardium edule.
Peringia ulvæ.
Hydrobia ventrosa.
Limapontia capitata.

Gasterosteus aculeatus.
Gobius microps.
Anguilla vulgaris

In the plankton only *Praunus flexuosus* was observed.

The shores of the inner basin are rocky, alternating with sandy bays. The only plants are *Potamogeton pectinatus*, and, in the fresher part, *Myriophyllum alterniflorum*.

In the salter parts the following are found:—

Gammarus duebeni.
Neomysis vulgaris.

Mya arenaria.

Gasterosteus aculeatus.
Gobius microps.
Anguilla vulgaris.

In the fresher parts the insects *Enallagma cyathigerum*, *Sympetrum striolatum*, Libellula sp. *Haliplus confinus* Steph. and *Gyrinus natator* occur along with Gammarus, Neomysis and Gasterosteus.

In the smaller lochs a fauna similar to that of the middle basin of Loch an Duin is in most instances to be found; the bryozoan *Valkeria uva* (L.) from Oban Alioter is the only species recorded in addition.

Notes on the Fauna.

Sarsia tubulosa.—This, the medusoid stage of the hydroid *Syncoryne sarsi*, appears to exist in three colour varieties (Kramp, 1927). The first, the "blue sarsia" of Kramp, with its tentacles and manubrium bright blue, occurs in Loch Strumore at a salinity of from $20\text{–}24^{\circ}/_{00}$. The "brown sarsia," the typical form, with the tentacles and manubrium brown, occurs in Oban a' Chlachain, at a salinity of $28\text{–}32^{\circ}/_{00}$. The "red sarsia" having the apical chamber and the tentacle bulbs a brilliant red, was not observed. A few specimens of the medusa were collected in Loch Obisary, but these were only noticed after they had been preserved so that their colour is not known. The hydroid was not obtained.

Procerodes ulvæ.—This animal occurs in numbers in an entirely unusual habitat. Whereas normally it lives on the shore at the mouths of streams and is subject to a daily alternation of fresh and salt water, in the lochs it lives all round the shores where the salinity hardly varies, and avoids the acid water of the stream mouths Its salinity limits are 5–33$^{\circ}/_{oo}$

Sphæroma hookeri.—This species of Sphæroma is only found in the Leodasay burn in North Uist, in a salinity varying between 3 and 30$^{\circ}/_{oo}$ according to the tide. In South Uist a few specimens were obtained in Loch Bee at a salinity of 33$^{\circ}/_{oo}$. These are the first Scottish records (Omer-Cooper and Rawson, 1935). Elsewhere in Britain it occurs in Norfolk, along the south coast of England, and in a few localities in Ireland, the nearest of which is Belfast Lough. *Sphæroma rugicauda* (Leach) is found in similar localities on the mainland, but up to the present is unrecorded from the Hebrides.

Idotea viridis.—This species seems limited in its occurrence by the distribution of Zostera and Ruppia among which it lives. In North Uist it is confined to salinities of 18–34$^{\circ}/_{oo}$. Elsewhere it occurs occasionally in the sea.

Neomysis vulgaris.—In North Uist Neomysis appears to be confined, with the exception of the pools on Clachan Strand, to water of low salinity, 1·2–12$^{\circ}/_{oo}$. In other places it occurs in nearly fresh water. Scott (1894) records it from Loch Sinclair in Barra, where the water is fresh to the taste, from Loch Westray in Caithness (1890) where the density is given as 1000·4, and from Shetland (Scott and Duthie 1894). Tattersall (1920) and Standen (1922) give other instances, but no water analyses are available and it is likely that a small amount of salt is present. Other records show that Neomysis is capable of living and breeding in much greater salinities, as at Aberlady (Nicol, 1935), where it occurred in salt-marsh pools with a salinity from 30–38$^{\circ}/_{oo}$. It is also recorded from the open sea. At Oban nam Fiadh, however, where it has a free range of salinity, it is confined to the fresher region.

Peringia ulvæ —Peringia occurs in two varieties, a strand form with a smooth horn-coloured shell and very shallow sutures, often of large size up to 9 mm. long. This is the only variety found on the strands at Clachan, Vallay and Vallaquie, but it also occurs in small numbers in some of the lochs, such as Oban Irpeig, Loch Obisary and Loch an Duin. The second variety is dark reddish-brown, with a rough eroded shell and deep sutures; the erosion increases in the older part of the shell towards the tip, and irregular regeneration may take place. This variety only occurs in the lochs, sometimes along with the strand form. Its salinity range appears to be from 10–34$^{\circ}/_{oo}$.

Hydrobia ventrosa occurs only in the lochs. It appears to prefer floating weed to any other habitat, although it occurs on all types of substratum except pure shell sand. Its salinity range is similar to that of Peringia.

Hydrobia jenkinsi.—The presence of this species of Hydrobia in the Outer Hebrides is extremely interesting, for it seems only lately to have been introduced into the mainland of Scotland. In 1903 it was reported from Perthshire, and Ellis (1925) states that it had not spread out of the valley of the Tay. Now, however, it occurs in Loch Leven, and also in a salt-marsh at the head of the Solway Firth. Yet in the Hebrides it is well established, occurring also in Benbecula and South Uist. It is without a larval dispersing stage and is viviparous so that it is quite conceivable that a single individual introduced into a new locality might in the course of time colonise it. This, then, may be an instance of dispersal by migrating birds, and it is noteworthy that the Tay, the Solway and the Hebrides are all on the route of migrating geese. *Hydrobia jenkinsi* has recently begun to colonise fresh water. In the Hebrides it is in an excellent position to do so, yet in Oban nam Fiadh and Loch Strumore, where fresh-water streams flow in, it only occurs for a few yards upstream, and is found in none of the fresh-water lochs. In one burn only at Sollas, on the northern machair land, is *Hydrobia jenkinsi* found in fresh water. The burn in question flows directly on to Vallay Strand. Pantin (1931) has shown how calcium influences the adaptability of *Procerodes ulvæ* to changing salinities, and Ellis (1933) has obtained similar results for *Nereis diversicolor*, in water of reduced salinity. It seems possible that the low calcium content of the water of the Hebrides may prevent Hydrobia from gaining a footing in fresh water except on the machair land where the presence of shell sand in the soil must raise the calcium content of the water. The salinity range of the species in North Uist is from 0·2–23°/₀₀.

Alderia modesta normally inhabits salt-marsh pools of fairly high salinity, and might be expected to occur in numbers in some of the lochs. A few were obtained from Oban na Curra and Oban a' Chlachain, but it occurs in much larger numbers in the small pools of a salt-marsh at Langass and on Clachan and Vallaquie strands where it was breeding in enormous numbers in July and August. On the strands the salinity at high tide cannot usually be much below that of normal sea water, but during the greater part of the twenty-four hours the surface of the sand may be moistened by rain or drainage water from the land, so that a brackish habitat is produced.

Limapontia capitata and *L. depressa.*—Both these species occur in the lochs among floating weed, but neither is abundant. Elsewhere (Nicol,

1933) *L. depressa* appears to prefer a semi-terrestial habitat and it was interesting to find, in corroboration of this observation, in a small muddy creek opening into Loch Eport and only filled by the tide for a few hours each day, thousands of this nudibranch living and breeding among *Vaucheria litorea* which covered the surface of the mud.

Mytilus edulis.—Mussels are by no means common round the coasts of North Uist, but do occur in some of the lochs in considerable numbers. Wherever they occur they show a reduction in size and a thinning and erosion of the shell similar to that shown by mussels in the Baltic. Table XIII gives the maximum size and weight of Mytilus at different salinities:

TABLE XIII.

	Salinity $^{\circ}/_{oo}$.	Maximum Length	Maximum Weight.
Loch an Duin	27 0	84 mm.	19 9 g.
Loch Strumore	22 4	38 ,,	1·2 ,,
Loch Obisary	12 2	25 ,,	0 5 ,,
Loch Obisary	9·8	17 ,,	0 2 ,,

Macoma balthica shows a similar dwarfing and erosion by the peaty water. It also shows that salinity is often not the most important factor in determining distribution in brackish water. Macoma occurs between the salinities of 10 and 34$^{\circ}/_{oo}$, yet is by no means common in the lochs, its occurrence there being rather dependent on a suitable type of bottom.

Gasterosteus aculeatus.—The three-spined stickleback occurs in its two forms var. *trachurus* and var. *gymnurus*; the former, with plates on the sides of the body and a flange on the mid-lateral line on each side of the tail, being found in brackish water, and the latter, without these characteristics, occurring in fresh water. Both varieties are found in the lochs. In July most of the adult fish have disappeared at any rate from the littoral zone, but those taken from Loch an Duin and Loch Strumore belong to var. *gymnurus.* Large individuals from Oban a' Chlachain belong to var. *trachurus*; small individuals in the same loch were of both kinds, in all other lochs they were unarmoured. M'Intosh mentions rough-tailed sticklebacks as occurring in the island.

DISCUSSION.

The large number of species occurring in the North Uist lochs, compared with the number in other brackish-water regions in Britain, is at first sight surprising. The greater part of the fauna, however, consists of marine species confined to a few lochs. When one considers the large area of water, the small degree of dilution and particularly the variety of substratum available in the lochs, the variety of species is more easily understood. For many animals the nature of the substratum appears to be at least as important as the salinity in controlling distribution. Examples of this are to be found in *Macoma balthica*, and to a lesser extent, *Corophium volutator*, which only occur when the bottom is of a particular consistency. A second reason for the large number of species lies possibly in the constancy of the salinity in the salter lochs. Unfortunately, data are only available for the months of June and July, but the close similarity between the salinity figures for 1933 and 1935, despite considerable differences in the spring rainfall, suggests that at no time are there great variations. The absence of streams of any size, and the spongy nature of the peat, probably reduce the chances of any sudden change of salinity in lochs such as Oban Irpeig, Oban a' Chlachain and Loch an Duin which are entered daily by the tide and where the greater part of the marine fauna is to be found.

In the lochs of North Uist the minimum number of species occurs at a salinity of 5°/₀₀. This low figure is due to the greater adaptability of the marine fauna, allowing it to tolerate a much greater change in salinity than can the less adaptable fresh-water fauna. This phenomenon has also been shown to occur in the Randersfjord by Johanssen (1918) and in the Baltic by Remane (1934). Fig. 6 shows the total numbers of species in the lochs in relation to salinity compared with the distribution of molluscs in the Randersfjord. Fig. 7 shows the distribution of the marine, brackish, fresh-water and euryhaline species in relation to salinity. The rapid drop in the number of marine species between the salinities of 35 and 25°/₀₀ and the complete disappearance of the members of the fresh-water fauna at 5°/₀₀ is well shown. The maximum number of brackish-water species occurs at a salinity of 25°/₀₀.

A detailed comparison with Randersfjord (Johanssen, 1918) indicates that in North Uist there is some factor present which narrows the salinity range of many species. This is shown in the following table, in which are compared the minimum salinities of the water in which some of the common molluscs are found to occur:—

TABLE XIV.

	Randersfjord. Salinity °/₀₀.	North Uist Salinity °/₀₀.
Mytilus edulis	5–6	10–11
Musculus discors . .	15–20	28–29
Peringia ulvæ . . .	1–2	10–11
Hydrobia ventrosa . . .	5–6	10–11
Littorina saxatilis var. *tenebrosa*	5–6	10–11
Macoma balthica . . .	5–6	10–11
Mya arenaria	5–6	5–6

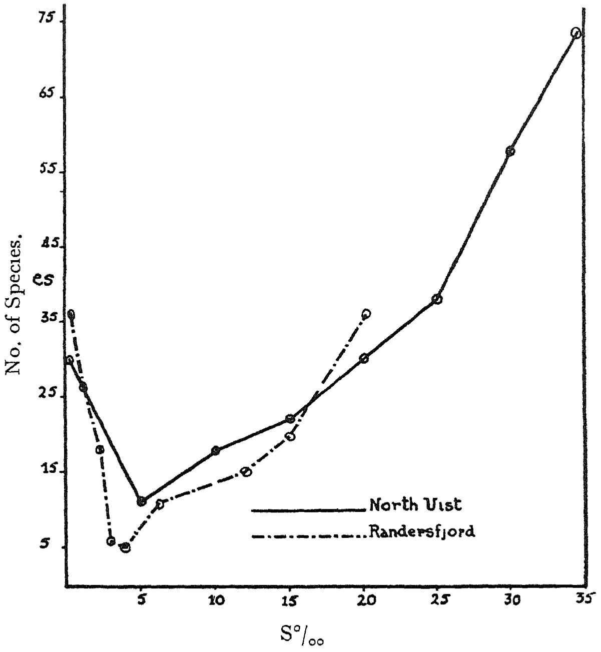

FIG. 6.—To show the relationship of the number of species to salinity in North Uist and in the Randersfjord (molluscs only).

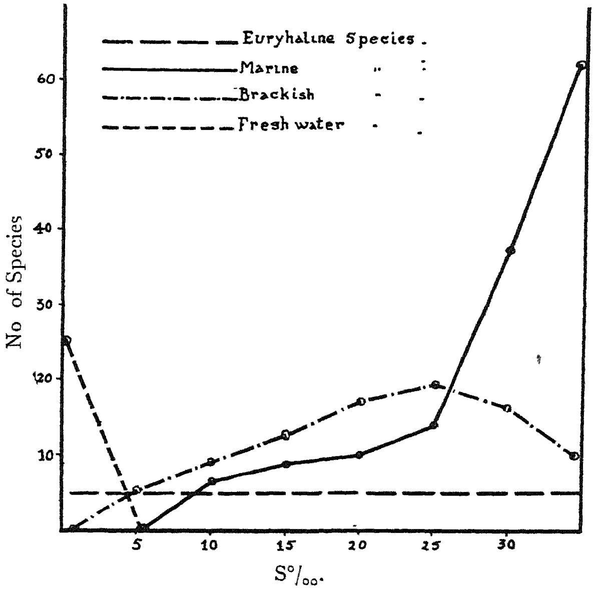

FIG. 7.—To show the distribution of euryhaline, marine, brackish and fresh-water species at different salinities.

This discrepancy suggests a chemical difference other than salinity between the brackish water of Randersfjord and of the North Uist lochs, something which enables animals to penetrate into lower salinities in the one than in the other. The first difference to suggest itself is a calcium deficiency effecting the permeability of the cell-wall. Ritchie (1934), however, has shown that *Procerodes ulvæ* inhabits the brackish mouths of streams having a calcium content as low as 5 mg. per litre. Assuming that the sea water is the only source of calcium in the lochs, at a dilution of even $1^{\circ}/_{\circ\circ}$ salinity, then 12 mg. calcium would still be present. On the face of this it is unlikely that calcium alone is a limiting factor in any brackish water, but until more is known of the part played by calcium and other constituents of sea water in osmotic control, and the quantity required by different species, the question must remain purely speculative.

SUMMARY.

The brackish-water lochs of North Uist lie at different levels and are influenced in varying degrees by the tidal inflow. All intermediate conditions between lochs entered by every tide and having a salinity of over $30^{\circ}/_{\circ\circ}$, and those entered by only the highest spring tides, with a salinity of 2 or $3^{\circ}/_{\circ\circ}$, are to be found. The hydrogen ion concentration of the fresh water on the island is about pH 5·4, but that of the brackish water is high, varying from 7·8 to 9·9. The alkali reserve is usually below that of sea water and is often below 0·0010 N.

The fauna of the lochs is varied; 59 marine, 24 fresh- and 25 brackish-water species were found as well as 5 euryhaline forms such as the salmon, which are equally at home in any salinity.

Graphs have been prepared to show the distribution of the animals at different salinities, and a comparison is made with the Baltic and elsewhere.

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(*Issued separately September* 11, 1936.)

XIII.—Cytological Studies on the Reproductive Organs. Chromosome Behaviour in the Male Grey Squirrel (*Sciurus carolinensis leucotus*). By **P. C. Koller,** D.Sc., Ph D , Institute of Animal Genetics, University of Edinburgh. *Communicated by* Professor F. A. E Crew, M.D. (With Twelve Figures)

(MS received June 26, 1936. Read July 6, 1936)

INTRODUCTION.

THE grey squirrel (*Sciurus carolinensis leucotus* Gapper), which is a native of the North American continent, was introduced into England and Scotland quite recently and soon became of great economic importance in these countries. In his very excellent monograph, Middleton (1930) states that both sexes of the grey squirrel show diminished activity during the winter and while they do not hibernate in the true sense of the word, they probably sleep if the weather is severe. The breeding habits of the female were studied by Deanesly and Parkes (1933), who ascertained that the female definitely has a period of anœstrus during which the reproductive organs are atrophic. In the male, however, fully functional testes were found throughout the year (Allanson, 1933), and there were no histological changes which would indicate a regular period of quiescence.

The primary object of the present paper is to give an account of the chromosome morphology and behaviour in the grey squirrel during the breeding season; this will form a part of a comparative study covering the nuclear cycle during the whole year.

MATERIAL AND TECHNIQUE.

Three adult grey squirrels were obtained through the kindness of Dr A. S. Parkes, National Institute of Medical Research, London, to whom I am greatly obliged. The animals were killed on the 10th of April 1936, and they are referred to in the following part as A1, A2 and A3. The first two were fully grown and of about the same size, A3 was smaller and appeared to be much younger.

Small pieces of the testes were fixed in several fixative solutions, strong Flemming's giving the most satisfactory results. The sections were cut at a thickness of 15–20 μ and stained with Newton's gentian violet and iron hæmatoxylin.

The drawings were made with the aid of a Zeiss camera lucida, using an 1·4 apochr oil-immersion objective and × 20 comp. eyepiece. The scale of magnification is given for each figure.

CHROMOSOME NUMBER AND MORPHOLOGY.

Abundant dividing spermatogonial cells were found in the testes of all the three animals investigated, and the chromosome complement could be satisfactorily analysed. The diploid number was found to be 28, which is the lowest chromosome number among the Eutheria. The grey squirrel appears to be a very suitable material for cytological study, because the cells are large and the chromosome number is small.

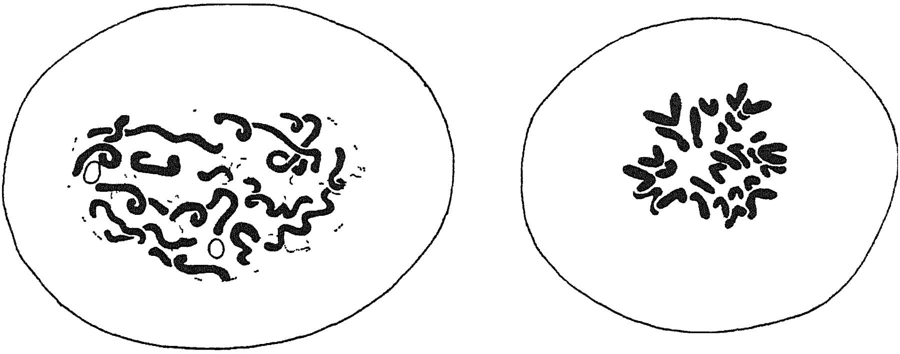

FIG. 1 —*a*, prophase of mitosis, showing the coiled chromosome threads; *b*, metaphase of mitosis. ×3500

Cross (1931), in a comparative study of the chromosomes of rodents, reported 48 as the diploid number in the American grey squirrel. It is unlikely, however, that such a great discrepancy could exist in the chromosome complement of the American grey squirrel and its acclimatised form, and it is probable that Cross was dealing with a different species or sub-species (*Sc. carolinensis carolinensis*). Further investigations are in progress to ascertain the chromosome number of the grey squirrel of present-day America

The nucleus of the dividing spermatogonial cells during the mitotic prophase is oval, and within the nucleus the chromosomes appear as twisted and coiled threads (fig. 1, *a*). It was demonstrated by Darlington (1935 *a*) that the external chromosome coiling at prophase is a remnant of the structure present at the telophase chromosomes of the previous division. Several chromatin aggregates may be identified in the nucleus; they are parts of the nucleolus, which breaks up into smaller elements at the early stages of mitotic prophase. It was found that all chromosomes exhibit the same degree of contraction and no precocious condensation was seen.

During metaphase the chromosomes are closely grouped together, the equatorial plate is rather small (fig 1, *b*). The members of the homologous chromosome pairs are distributed at random while the arrangement observed in Marsupials and described as partial somatic pairing (Koller, 1936) was not found. During mitotic metaphase no definite unequal chromosome pair could be identified, which indicates that the sex chromosomes are of the same length or nearly so.

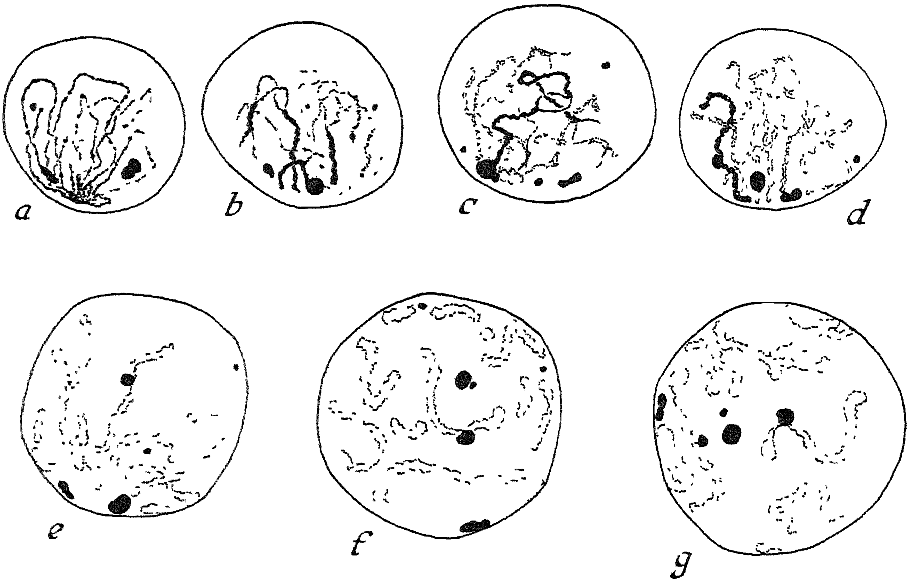

FIG 2.—Prophase of meiosis. *a*, leptotene; *b*, zygotene, *c*, pachytene, showing interlocking; *d*, "synizesis," *e g* "growth period" ×3500

CHROMOSOME BEHAVIOUR AND CHIASMA FREQUENCY DURING MEIOSIS.

The meiotic prophase commences by the appearance of fine chromosome threads (leptotene, fig. 2, *a*). They emerge from an area which lies usually near the periphery of the nucleus. Such arrangement of chromosome threads is very striking and similar to that found at late pachytene of various organisms and described by several cytologists as polarisation. The position of leptotene chromosome threads, however, is not the result of an attraction operating between the ends of chromosomes and the attraction pole, but is brought about by the chromosome arrangement at the end of previous division Several nuclear aggregates may be identified lying among the chromosome threads during the leptotene and the following zygotene. These aggregates are of various sizes and some of them have originated from the nucleolus, which is a permanent constituent of the nucleus. The association of homologous chromosomes (zygotene stage) usually begins terminally (fig. 2, *b*), and intercalary segments of varying size are often left unpaired for a long time.

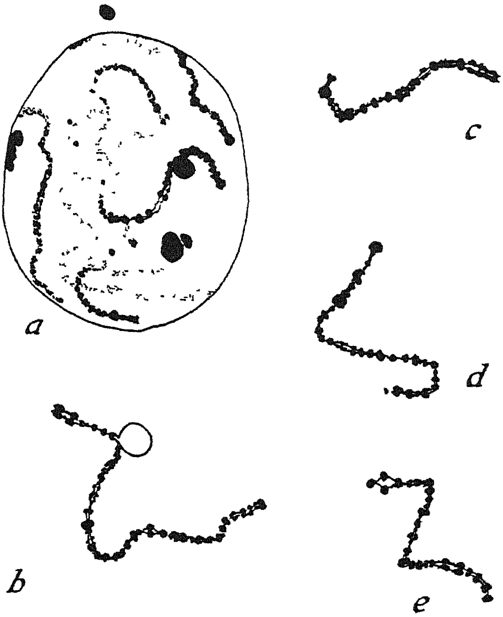

FIG. 3.—*a*, pachytene nucleus, showing the four (*b*, *c*, *d*, *e*) paired chromosome threads which have a characteristic structure × 4000

The pachytene is one of the longest stages of meiotic prophase, as indicated by the great number of primary spermatocytes showing the various phases of that stage. The association of the homologous chromosomes is completed and they are closely packed together (fig. 2, *d*). This stage is very often referred to as "synizesis" and found to be characteristic of several organisms, mostly animal. During the early pachytene three, sometimes four, paired chromosome threads can be recognised by their structure. One of them has a subterminal intercalary nucleolus (fig. 3, *a* and *b*); in the neighbourhood of that locus the chromosome is thicker and stains more readily than in the distal regions. Two other chromosomes have a large terminal or subterminal chromomere (fig 3, *c* and *d*). The fourth bivalent exhibits a very characteristic chromomere arrangement at one end (fig. 3, *e*).

During pachytene several nuclei were found with chromosomes interlocked, complete pairing being prevented (fig. 2, *c*). Besides the large nucleolus and smaller chromatin granules, another aggregate with double structure was encountered. By studying the behaviour of this particular nuclear constituent in the following stages of meiosis, it was identified as the precociously contracted XY bivalent.

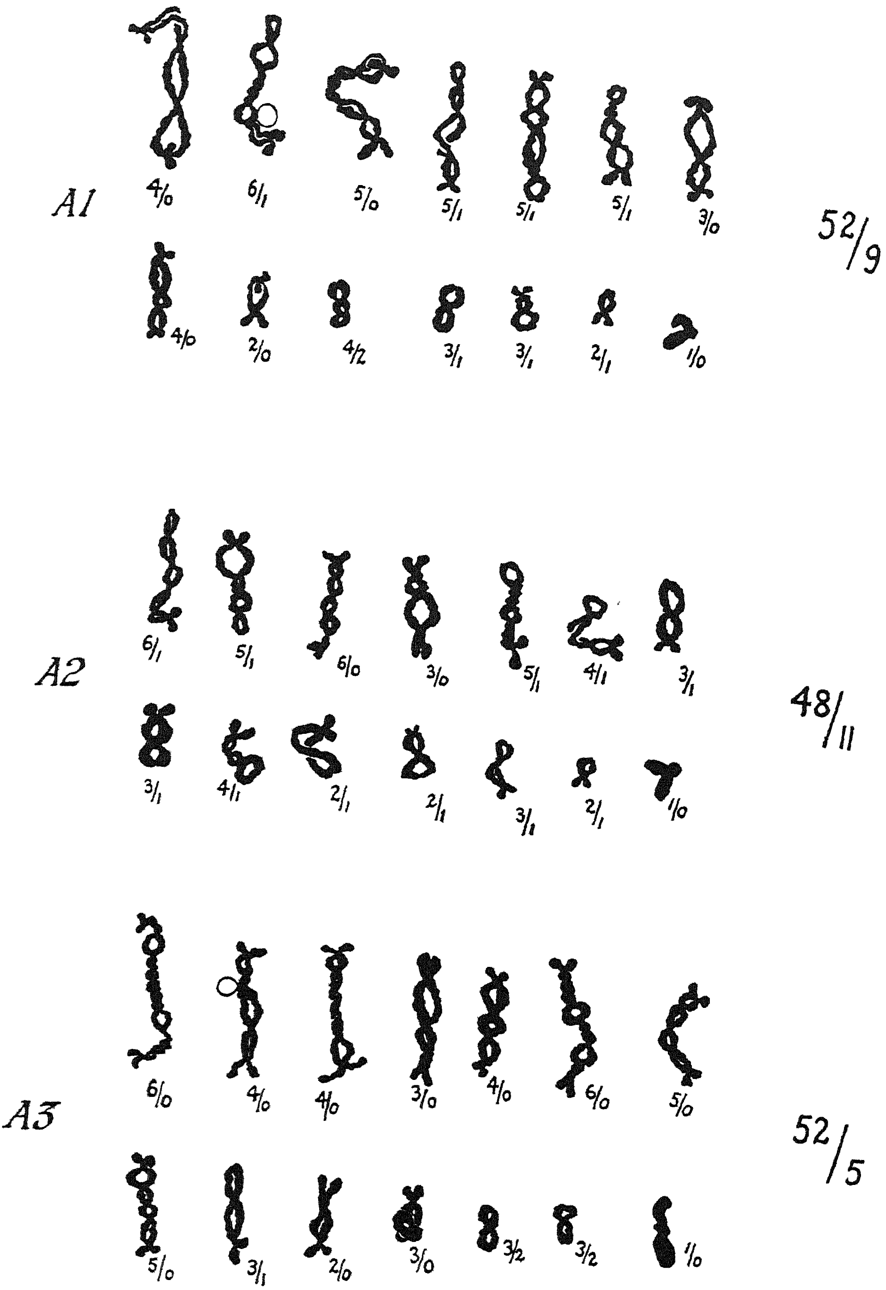

FIG. 4.—Mid-diplotene in A1, A2 and A3. The number of total and terminal chiasmata is given for each bivalent and for the three nuclei ×4000.

After "synizesis" the bivalents lose their staining capacity; they may stain to a certain extent with iron hæmatoxylin, but fail to do so by gentian violet. The nucleolus and other nuclear aggregates, however, stain as before. The volume of the nucleus is increased and the paired chromosome threads are now scattered (fig. 2, *e–g*). This stage may be described as the "growth" period.

The beginning of the diplotene is marked by the appearance of small loops in the lightly stained bivalents. The size and number of these loops increase as the bivalents become more visible and more stained. The nodes of chromosome pairs represent the chiasmata, *i.e.* the loci where exchange between partner chromatids has occurred. As the bivalents contract, they stain more deeply and hence their structure can be analysed in greater detail. During mid-diplotene (fig. 4) the number of chiasmata in the longer bivalents was found to be 4, 5 or 6. There is no direct relationship between the incidence of chiasmata and the length of the bivalents, but if the short bivalents are disregarded the mean number of chiasmata in a bivalent appears to be approximately proportional to its length. Several bivalents exhibit only two or three very small loops and the segments which lie between two adjacent chiasmata do not form open loops, but appear to be coiled. This structure of mid-diplotene bivalents is assumed to be the consequence of a specific method of pairing during pachytene, namely, "relational" coiling of the homologues round each other, which is responsible for genetical crossing-over (Darlington, 1935 *b*).

One or, occasionally, two bivalents were found during diplotene and diakinesis to contract before the others. One of them is the sex bivalent, showing a subterminal or terminal chiasma; the other belongs to the smaller chromosome group, and contains two or three chiasmata.

The chiasma frequencies of the three individuals A1, A2 and A3 during the various stages of meiosis are given in Table I.

It should be noted that the figures given for diplotene do not represent the initial frequencies, which is probably slightly higher. The distribution of chiasmata in the different bivalents indicates negatively correlated chiasma frequencies, *i.e.* some influence of one bivalent on chiasma formation in others within the same nucleus, irrespective of outside influence (Mather, 1936).

Though in diplotene the mean number of chiasmata per bivalent in A1 and A3 is almost the same (3·6 and 3·7 respectively), the deviation from the mean number of chiasmata per nucleus is much greater in A1 (Table II).

TABLE I.—THE CHIASMA FREQUENCY IN THREE SQUIRRELS

Animal	Stages of Meiosis	No. of Nuclei.	Number of Bivalents with						Total No. of Chiasmata.	Total No of Term Xta †	Mean No of Xta per Bivalent	Mean No of Xta per Nuclei	Term Coefficient.
			6Xta *	5Xta	4Xta	3Xta.	2Xta	1X					
A1	Diplotene	10	14	36	26	29	27	11	504	92	3·6	50·4	0·18
	Diakinesis	5	..	.	1	24	30	15	151	57	2·1	30	0·37
	Metaphase	10	..			5	56	79	206	31	1·4	20·6	0·44
A2	Diplotene	10	18	21	22	38	29	12	485	107	3·4	48·5	0·24
	Diakinesis	5	.	.	1	18	34	20	137	42	1·9	27	0·20
	Metaphase	5	..	.	.	3	28	39	104	41	1·5	20	0·39
A3	Diplotene	10	19	22	29	47	14	1	518	51	3·7	51·8	0·09
	Diakinesis	5	..		10	24	21	15	169	51	2·4	33	0·30
	Metaphase	10	..	.	6	24	68	42	274	87	1·9	27·4	0·31

* Xta indicates chiasmata. † Term. Xta indicates terminal chiasmata.

TABLE II.—THE DISTRIBUTION OF DIPLOTENE CHIASMA FREQUENCIES IN THREE SQUIRRELS

Animal.	Number of Xta per Nucleus.								Mean No. of Xta per Nucleus
	46	47	48	49	50	51	52	53	
A1	..	1	1	1	1	3	2	1	50·4
A2	1	2	3	..	3	1	..	.	48·5
A3	..	..	..		..	4	4	2	51·8

Use of χ^2 test shows that the possibility of such a difference in distribution arising by chance is very unlikely.

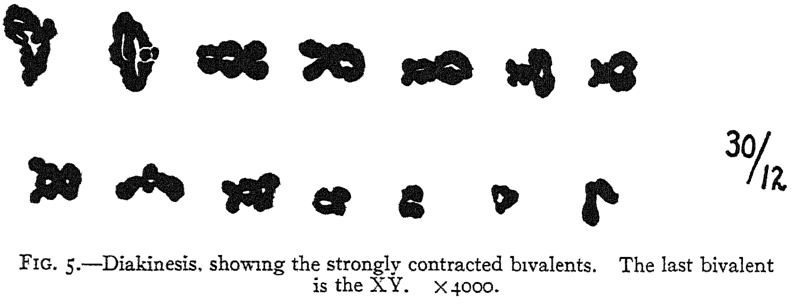
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FIG. 5.—Diakinesis, showing the strongly contracted bivalents. The last bivalent is the XY. ×4000.

The number of chiasmata is reduced during diakinesis and the bivalents are shorter and thicker, indicating a strong contraction (fig 5). The decrease in the number of chiasmata is due to terminalisation, brought about by forces of repulsion operating between centromeres (Darlington, 1932). The bivalents are frequently grouped together, indicating that the inter-chromosomal repulsion is weak. The nucleolar bivalent was not detected during diakinesis. The number of intra-nuclear aggregates is between 2 and 4; they may be of various sizes.

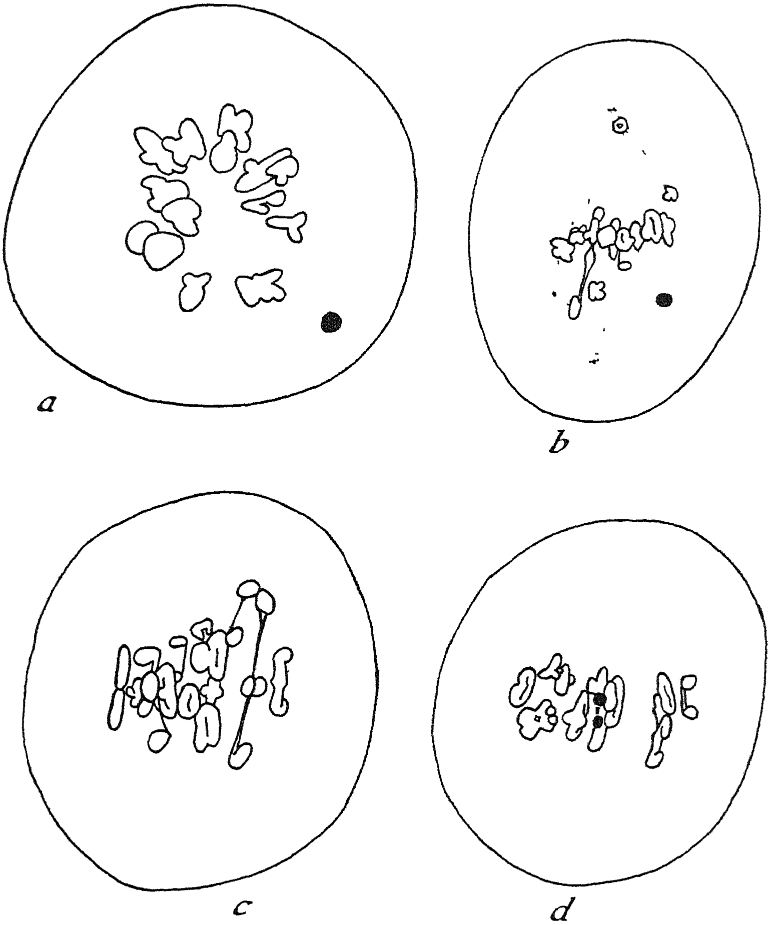

FIG. 6.—Metaphase of meiosis. *a*, polar view of bivalents and the chromatoidic body (black); *b*, side view, showing the small bivalents lying off the equatorial plate; *c*, metaphase from A2; and *d*, metaphase from A3. The sex bivalents are black. × 3500

During meiotic metaphase the bivalents are arranged in the equatorial plate. In polar view they are seen to form a more or less complete circle (fig. 6, *a*). Such an arrangement of bivalents during metaphase is very characteristic of the primary spermatocytes in the squirrel. The small bivalents may occasionally lie off the metaphase plates (fig. 6, *b*), but always remain within the spindle. The large round chromatoid body, which is a constant extra-nuclear constituent of the primary spermatocytes, can be identified clearly. In several cells the metaphase bivalents are smaller than in the other primary spermatocytes of comparable volume (fig. 6, *b* and *c*).

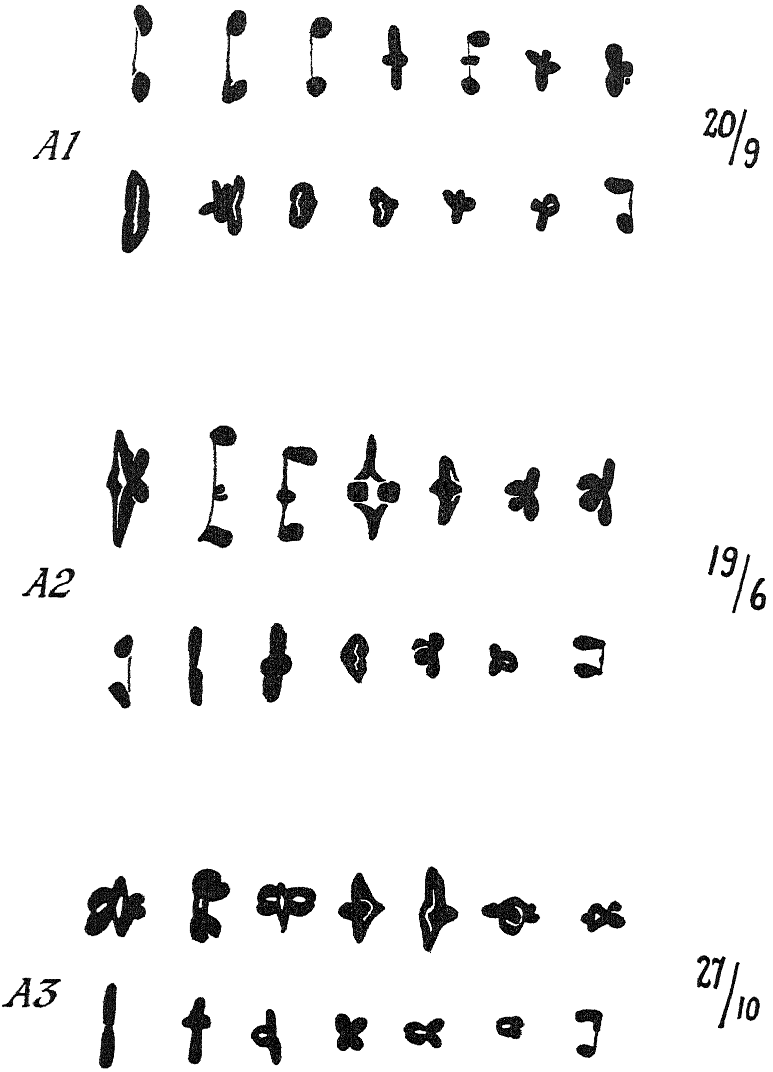

FIG. 7.—Metaphase bivalents in A1, A2 and A3. The number of total and terminal chiasmata is given. ×4000.

The types of bivalents during metaphase in animal A3 differ from those found in A1 and A2 in that A3 has 28 as a mean number of chiasmata per nucleus, while A1 and A2 have 20 only (fig. 7). It was seen in Table I that the diplotene chiasma frequencies are very similar in A1, A2 and A3, hence it is probable that there is a different type of terminalisation in A3, which is responsible for the observed deviation in the types of metaphase bivalents. In A1 the terminalisation of chiasmata from diplotene to metaphase is gradual, while in A3 it is very high from diplotene to diakinesis and becomes almost nil thereafter Another interesting phenomenon was found in animal A2, namely, the number of terminal chiasmata is slightly less at diakinesis than it is in diplotene. The terminalisation coefficient in the three animals (calculated by dividing the number of terminal chiasmata by the total number of chiasmata) is illustrated in fig. 8. The ultimate cause of the discrepancies between the terminalisation coefficients may be either genetical or environmental (age.)

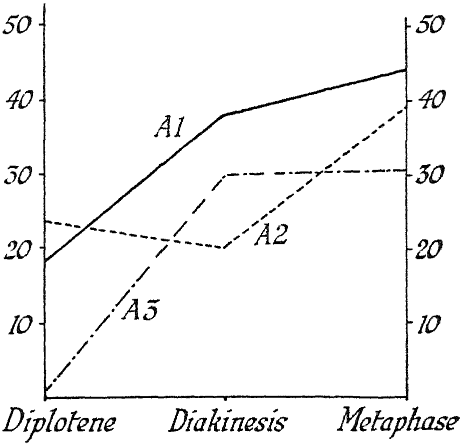

FIG. 8 —The terminalisation coefficient in A1, A2 and A3.

The sex bivalent could easily be identified during diplotene owing to the precocious contraction or condensation. During metaphase, however, it is difficult to detect because it lies among the autosomal bivalents (fig. 6, *d*), and, furthermore, there is no great difference between the length of the X and the Y. The sex chromosomes are commonly associated during the meiotic metaphase by either a terminal or subterminal chiasma. The type of XY during metaphase was invariably found to be asymmetrical, indicating an obligatory pre-reduction due to the position of the centromere in relation to the differential segment. From the type of the XY bivalent one may assume that the centromere is localised either in the pairing segment next to the differential segment, or, more probably, at the locus where the two regions meet. The greater part of the sex chromosomes is composed of the differential segment (fig. 9).

Several binucleate and multinucleate spermatocytes were encountered in animal A2 (fig. 10). They are produced by an arrest of cytokinesis and form a well-demarcated group in the seminiferous tubules, which suggests a common origin. Every stage of meiosis was found in such cells. The bivalents of the different complements remain separated during the entire nuclear cycle. The metaphase chiasma frequencies of three binucleate spermatocytes are given in Table III.

TABLE III.—CHIASMA FREQUENCIES DURING METAPHASE IN THREE BINUCLEATE SPERMATOCYTES.

Cell.	Chromosome Complement.	Number of Bivalents with 3Xta	2Xta.	1X	Total and Term. Xta.
1	*a*	. .	4	8	18/3
	b	1	5	6	19/6
2	*a*	.	5	9	19/4
	b		7	7	21/6
3	*a*	2	3	9	18/5
	b	1	6	7	20/7

During the first meiotic anaphase the bivalents segregate simultaneously. In a few spermatocytes a bivalent may remain connected by

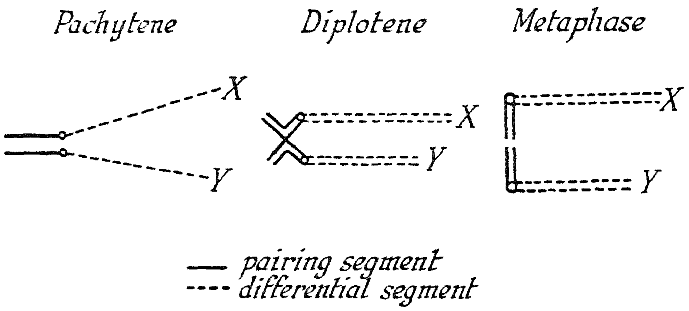

FIG. 9.—The structure of the sex bivalent during meiosis.

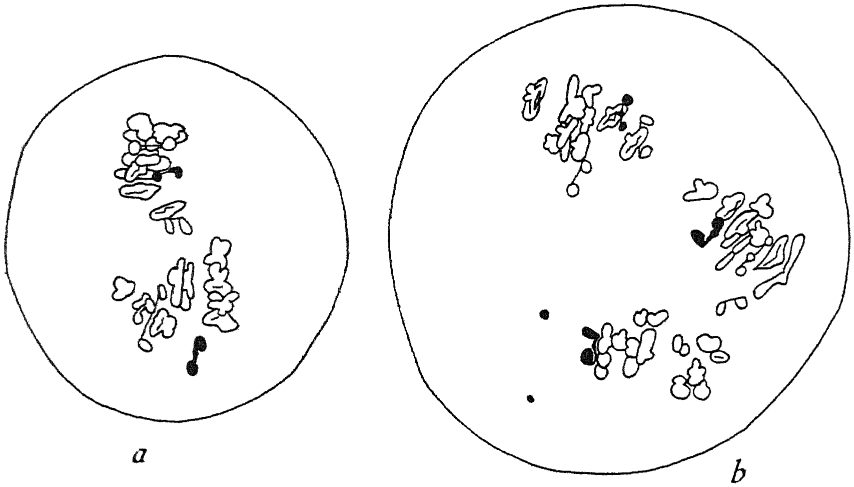

FIG. 10.—*a*, binucleate; and *b*, multinucleate spermatocytes with three separate chromosome complements. The sex bivalents are black. × 3000.

a more or less thin bridge (fig. 11, *a*) Such a bridge may be assumed to indicate the presence of an inversion, since crossing-over in an inverted segment would give rise to chromatid bridge or dicentric chromatid and fragments. However, the complete absence of fragments and the extremely rare incidence (less than 1 per cent.) of such chromatid bridges suggests that this configuration is due not to an inversion but rather to a high number of chiasmata at that particular bivalent delaying the separation of the associated chromosomes.

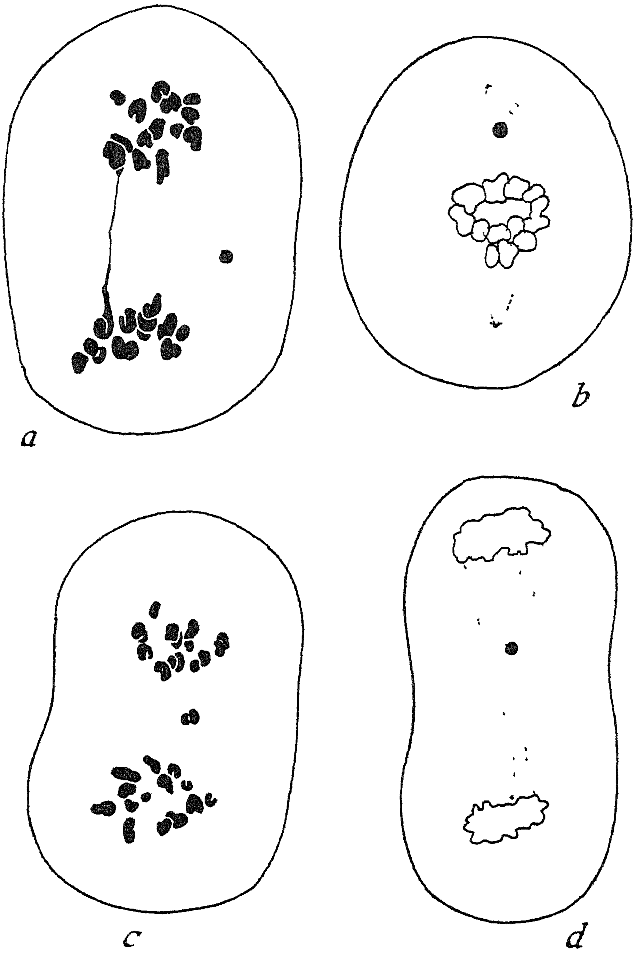

FIG. 11.—*a*, first meiotic anaphase, showing a chromatid bridge; *b*, second metaphase, the chromosomes form a circle; *c*, second meiotic anaphase with a lagging chromosome; *d*, the "inter-chromosomal" spindle with the chromatoid body ×3500.

The large chromatoid body usually lies at the middle of the spermatocytes, and is carried by the cytoplasmic movement towards one pole during anaphase without division.

The second meiotic metaphase follows anaphase immediately. The chromosomes are commonly arranged in a circle at the equatorial plate and show the double structure. Relatively few secondary spermatocytes showing metaphase were encountered, which indicates that this stage is short. In a few instances lagging of one or two chromosomes was observed (fig. 11, *c*). During the second anaphase the inter-chromosomal spindle persists for a long time and includes the chromatoid body (fig. 11, *d*) The spindle can frequently be seen even at the time when the chromosomes are already fused and the cell-membrane is formed between the two daughter spermatids (fig. 12, *a*, *b*). In the middle of the inter-chromosomal spindle or in the region of the developing new cell membrane, the spindle fibres thicken and stain very well.

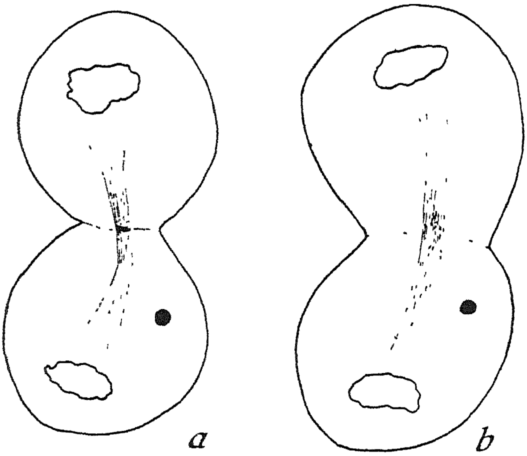

FIG 12.—Spermatids (*a* and *b*), showing the inter-chromosomal spindle during cytokinesis ×3500

The spermatids are of two kinds in respect of the sex chromosomes, but these types cannot be distinguished morphologically owing to the irregular variation in the number and size of the intra- and extra-nuclear chromatin aggregates.

SUMMARY.

1. The diploid chromosome number of the grey squirrel (*Sciurus carolinensis leucotus* Gapper) is 28. The chromosomes differ in size and shape; the longest is 4·5 μ and the shortest is about 1 μ.

2. Four paired chromosome threads can be identified by their permanent structure at pachytene.

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